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Math.
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For GE-2nd sem students

GE-2T Algebra, Inverse Matrix

P.I

27/4/20

p1

Inverse of a matrix: A square matrix A of order n is said to be invertible if there exists a matrix B s.t. $AB = BA = I_n$, then B is said to be an inverse of A .

NB. ① An invertible matrix has a unique inverse.

Proof: Let A be an invertible matrix of order n . If possible let B and C be two inverses of A . Then

$$AB = BA = I_n, AC = CA = I_n.$$

We have $C(AB) = (CA)B$ since Multiplication is associative.

$$\Rightarrow C I_n = I_n B \Rightarrow C = B.$$

This Proves that A has a unique inverse.

② A square matrix A is said to be non-singular if $\det A \neq 0$. and singular if $\det A = 0$.

③ An $n \times n$ matrix A over a field F is invertible if and only if A is non-singular.

Proof: Let A be an $n \times n$ invertible matrix

Then there exists an $n \times n$ matrix B s.t.

$$AB = BA = I_n.$$

$$\text{we have } \det(AB) = \det(I_n) = 1 \text{ or.}$$

$$\det A \cdot \det B = 1.$$

This implies that $\det A \neq 0$.
and therefore A is non-singular.

Conversely: Let A be non singular matrix of order n . Then $\det A \neq 0$.

$$\text{Now for an } n \times n \text{ matrix } A, A \cdot \text{adj } A = \text{adj } A \cdot A = \det(A) I_n$$

$$\text{Since } \det A \neq 0 \quad A \cdot \frac{1}{\det A} (\text{adj } A) = \frac{1}{\det A} (\text{adj } A) \cdot A = I_n.$$

From the definition of an inverse it follows that $\frac{1}{\det A} (\text{adj } A)$ is the inverse of A . Hence A is invertible.

$$\text{NB: } (CA)^{-1} = A^{-1}C^{-1} \text{ But } (AB)^{-1} = B^{-1}A^{-1}.$$

Ex-2 Prove $(AB)^{-1} = B^{-1}A^{-1}$.

Let A, B be $n \times n$ matrices, since A and B are invertible $\Rightarrow$ then A^{-1}, B^{-1} exist and
 $AA^{-1} = A^{-1}A = I_n$ and $BB^{-1} = B^{-1}B = I_n$. & $|B| \neq 0$.

we have $\det(AB) = \det A \cdot \det B \neq 0$.

This shows AB is non-singular and hence it is invertible.

$$\begin{aligned} \text{Now } (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n \\ \text{and } (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = (B^{-1}I_n)B = B^{-1}B = I_n. \\ \text{Hence } (AB)(B^{-1}A^{-1}) &= (B^{-1}A^{-1})(AB) = I_n. \end{aligned}$$

it follows that $B^{-1}A^{-1}$ is the inverse of AB .

If A be a matrix (non-singular)

Then $A^{-1} = \frac{1}{|A|} \text{adj } A$. which is the reciprocal of A .

NB: If the inverse of the matrix A exists, then A is called invertible.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and let Cofactor of $a_{ij} = A_{ij}$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Ex. Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

Let Here A_{11} Cofactor of 1 = $36 - 25 = 11$

Cofactor $2(A_{12}) = \begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -7$, $A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2$, $A_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -9$

$A_{22} = 12 - 3 = 9$, $A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -3$, $A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1$, $A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2$

$A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$ Hence $\text{Adj } A = \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix} = 3$ hence $A^{-1} = \frac{1}{3} \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

Inverse of Matrix

p3

Ex. Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

Sol: We know $A^{-1} = \frac{1}{|A|} \text{adj} A$. $\text{if } |A| \neq 0$
 A^{-1} exists
find $|A|$ first

$\text{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$ where A
 T represents
 Transpose
 Transformation
 of rows into columns

where

A_{11} is cofactor of element 1 in Matrix $A = \begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = 36 - 25 = 11$

Similarly $A_{12} = \begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -7$

like wise $A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 5 - 3 = 2$

A_{21} cofactor of 1 in the second row $= - \begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -9$

So $A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 9$, $A_{23} = - \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -(5 - 2) = -3$

$A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$; $A_{32} = - \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -5 + 3 = 2$

$A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

Hence $\text{adj} A = \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

$$\det A = |A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix} \quad \begin{array}{l} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - R_1 \end{array}$$

$$\sim \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 9 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 \\ 3 & 9 \end{vmatrix} = 9 - 6 = 3.$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{3} \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & 2 \\ 2 & -3 & 1 \end{bmatrix}$$

check $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans.

Ex. Find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{bmatrix}$

Sol. $\det A = |A| = 8 \begin{vmatrix} 8 & 4 \\ 2 & 8 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix} + 2 \begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} = 392$

$|A| \neq 0$ so inverse of A exists

Matrix of Cofactors of $A = \begin{bmatrix} \begin{vmatrix} 8 & 4 \\ 2 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix} & +\begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} & +\begin{vmatrix} 8 & 2 \\ 1 & 8 \end{vmatrix} & -\begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} & -\begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 8 & 4 \\ 2 & 8 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 56 & -12 & -4 \\ -28 & 62 & -12 \\ 0 & -28 & 56 \end{bmatrix}$$

$\therefore \text{Adj} A = \text{Transpose of matrix of Cofactors of } A$

hence $A^{-1} = \frac{1}{392} \begin{bmatrix} 56 & -28 & 0 \\ -12 & 62 & -28 \\ -4 & -12 & 56 \end{bmatrix}$

$$= \begin{bmatrix} \frac{56}{392} & \frac{-28}{392} & 0 \\ \frac{-12}{392} & \frac{62}{392} & \frac{-28}{392} \\ \frac{-4}{392} & \frac{-12}{392} & \frac{56}{392} \end{bmatrix}$$

Ans.