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GE-2 (math)

Algebra (unit-3)

System of linear equations

A linear system of m equations in n unknowns $x_1, x_2, \cdots x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{———} (1)$$

This system is called linear because variables appear in 1st power, where $a_{11}, a_{12}, \cdots a_{mn}$ are called co-efficients. If all the b_j are zero, then (1) is called homogeneous system. If at least one b_j is not zero then (1) is called a nonhomogeneous system. If the system (1) is homogeneous it has at least the trivial solution $x_1 = 0, \cdots x_n = 0$. A solution of (1) is a set of numbers $x_1, \cdots x_n$ that satisfies all m equations. A solution vector of (1) is a vector x whose components form a solution of (1)

(1) may be written as a single vector equation

$$\underline{AX = B} \quad \cdots \quad (2)$$

where the co-efficient matrix $A = [a_{jk}]$ is the $m \times n$ matrix where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ Column vectors are X and b.

The matrix $\tilde{A} = (A, b) = [A \vdots b] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & \vdots & b_1 \\ \cdots & \cdots & \cdots & \vdots & \vdots \\ & & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$ is called

the augmented matrix of the system (1)

If $m = n = 2$ two unknown in two equation

ie $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

for 3 equations in 3 unknown. is

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$ etc

Two types of linear equation (i) Consistent ~~(ii) inconsistent~~ (ii) inconsistent.

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$

Here $AX = B$.

Let $C = [A, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$

then C is called augmented matrix

a) For Consistent equations; if Rank A = Rank C
(i) unique solution if Rank A = Rank C = n
where n = number of unknown.

b) Inconsistent. If Rank A ≠ Rank of augmented matrix C

Echelon form of a matrix:

i) Every row of A which has all its entries 0 (zero) ~~row is equal to 1 (Unity) [may not be]~~ occurs below every row which has a nonzero entry.

ii) The number of zeros before the 1st non zero ~~in~~ element in a Row is less than the number of such zeros in the next row.

Important: The rank of a matrix in a Echelon form is equal to the number of NON Zero rows of the matrix

Example of Echelon form Matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

we may conclude Rank of A = 2.

Elementary transformations.

1. Interchanging any two rows (two columns) of the matrix $R_i \leftrightarrow R_j$
2. Multiplication of the elements of any Row (column) by a non zero scalar quantity of the matrix mR_i (m_i const)
3. Addition of constant multiplication of the elements of any Row to the corresponding elements of any other row is denoted by $R_i \rightarrow R_i + kR_j$. k is constant.

Ex ①
Transform the matrix A to row reduced upper triangular echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix}$$

Now by $R_3 \longrightarrow R_3 + 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Example of Row-reduced echelon matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

EX: Express the following system of linear equation as matrix equation

$$\left.\begin{aligned} 2x+3y-z &= 1 \\ x+y+2z &= 2 \\ 2x-y+z &= 3 \end{aligned}\right\}$$

Soln: The system of equations can be written as

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ or } AX = B$$

where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Ex. Reduce the following matrix to a echelon matrix

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \text{ by } \tfrac{1}{2}R_1 \sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\text{by } \begin{matrix} R_2 - 3R_1 \\ R_3 - 5R_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \text{ by } \tfrac{1}{2}R_2 \sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\text{by } \begin{matrix} R_3 - 2R_2 \\ R_4 - 3R_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \text{ by } R_{34} \text{ interchanging row 3} \rightleftarrows \text{row 4} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Now by } \tfrac{1}{2}R_3 \sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_1 - 2R_3 \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is row echelon matrix its has 3 - non zero row so its rank = 3.

Ex. Find the solution for linear system

$$\left.\begin{aligned} x_1 + 2x_2 - x_3 &= 1, \\ 2x_1 + x_2 + 4x_3 &= 2 \\ 3x_1 + 3x_2 + 4x_3 &= 1 \end{aligned}\right\}$$

Sol^n: Performing the row operation of the augmented Matrix from the given linear equations. equation in vector form (Matrix form)

$$AX = B \text{ ie } \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right] \text{ by } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{array}\right]$$

$$\text{by } -\tfrac{1}{3}R_2 \;\&\; R_3 - R_2 \sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{matrix} R_1 + R_3 \\ R_2 + 2R_3 \end{matrix} \sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \text{ by } R_1 - 2R_2 \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system is equivalent to $x_1 = 7$, $x_2 = -4$ & $x_3 = -2$

Which shows that the system has unique solutions