Semester - VI

Course Type - DSE - 4

Course Title: DSE - 4: Mathematics Modelling

Topic: L T Application

References: Dr. Bej & Mukherjee Book.

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Example 4. Solve $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=e^{-t}$, given $y(0) = y'(0) = 0$.

Solution :

The given equation may be taken as

$$y'' + 3y' + 2y = e^{-t} \quad \text{... (1)}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$

Taking Laplace transform on both sides of (1) we get

$$L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{e^{-t}\}$$

or, $s^2L\{y\} - sy(0) - y'(0) + 3[sL\{y\} - y(0)] + 2L\{y\} = \frac{1}{s+1}$

or, $(s^2 + 3s + 2)\, L\{y\} = \frac{1}{s+1}$

or, $L\{y\} = \frac{1}{(s+1)(s^2+3s+2)} = \frac{1}{(s+2)(s+1)^2}$

or, $L\{y\} = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$

or, $y = L^{-1}\left\{\frac{1}{s+2}\right\} - L\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$= e^{-2t} - e^{-t} - e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}$, using shifting theorem

$= e^{-2t} - e^{-t} - e^{-t}.t$

Hence the required solution is $y = e^{-2t} - (1 + t)e^{-t}$

Example 5. Solve $(D^2 + 2D + 1)y = 3te^{-t}$, given $y = 4$, $Dy = 2$ when $t = 0$.

Solution :

The equation is $y'' + 2y' + y = 3te^{-t}$ ... (1)

with the initial conditions $y(0) = 4$ and $y'(0) = 2$

Taking Laplace transform on the both sides of (1) we get

$$L\{y''\} + 2L\{y'\} + L\{y\} = 3L\{te^{-t}\}$$

or, $s^2L\{y\} - sy(0) - y'(0) + 2[sL\{y\} - y(0)] + L\{y\} = -3\frac{d}{ds}[L\{e^{-t}\}]$

or, $(s^2 + 2s + 1)\, L\{y\} - 4s - 2 - 8 = -3\frac{d}{ds}\left(\frac{1}{s+1}\right)$

or, $(s^2 + 2s + 1)\, L\{y\} = \frac{3}{(s+1)^2} + (4s + 10)$

or, $L\{y\} = \frac{4s+10}{(s+1)^2} + \frac{3}{(s+1)^4} = \frac{4(s+1)+6}{(s+1)^2} + \frac{3}{(s+1)^4}$

$= \frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}$

$\therefore \quad y = L^{-1}\left\{\frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}\right\}$

$= e^{-t}L^{-1}\left\{\frac{4}{s} + \frac{6}{s^2} + \frac{3}{s^4}\right\}$, using shifting theorem

$= e^{-t}\left(4 + 6t + \frac{1}{2}t^3\right)$

Hence, $\quad y = \left(4 + 6t + \frac{1}{2}t^3\right) e^{-t}$

which is the required solution.

Example 6. Solve $ty'' + y' + 4ty = 0$, if $y(0) = 3$, $y'(0) = 0$.

Solution :

The equation is $ty'' + y' + 4ty = 0$ ... (1)

with the initial conditions $y(0) = 3$ and $y'(0) = 0$

Taking Laplace transform on both sides of (1) we get

$$L\{ty''\} + L\{y'\} + 4L\{ty\} = 0$$

or, $(-1)^1\frac{d}{ds}L\{y''\} + L\{y'\} + 4(-1)^1\frac{d}{ds}L\{y\} = 0$

$$\text{or,}\quad -\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] + sL\{y\} - y(0) - 4\frac{d}{ds}L\{y\} = 0$$

$$\text{or,}\quad -\frac{d}{ds}[s^2L\{y\} - 3s] + sL\{y\} - 3 - 4\frac{d}{ds}L\{y\} = 0 \qquad \cdots (2)$$

Let $L\{y\} = \phi$,

then, (2) reducess to

$$-\frac{d}{ds}(s^2\phi - 3s) + s\phi - 3 - 4\frac{d\phi}{ds} = 0$$

$$\text{or,}\quad -\left[s^2\frac{d\phi}{ds} + 2s\phi - 3\right] + s\phi - 3 - 4\frac{d\phi}{ds} = 0$$

$$\text{or,}\quad (s^2 + 4)\frac{d\phi}{ds} + s\phi = 0$$

$$\text{or,}\quad \frac{d\phi}{\phi} + \frac{1}{2}\cdot\frac{2s}{s^2+4}ds = 0$$

Integrating we get

$$\log\phi + \frac{1}{2}\log(s^2 + 4) = \log c$$

$$\text{or,}\quad \phi = \frac{c}{\sqrt{s^2+4}}$$

$$\text{So,}\quad L\{y\} = \frac{c}{\sqrt{s^2+4}}$$

$$\text{Therefore,}\quad y(t) = L^{-1}\left\{\frac{c}{\sqrt{s^2+4}}\right\} = cL^{-1}\left\{\frac{1}{\sqrt{s^2+2^2}}\right\} = cJ_0(2t) \qquad \cdots (3)$$

where $J_n(n)$ is the Bessel function of order n and $L\{J_0(at)\} = \dfrac{1}{\sqrt{s^2+a^2}}$

Putting $t = 0$ in (3), we get

$y(0) = cJ_0(0)$ or, $3 = c$ [since $J_0(0) = 1$]

Hence from (3) we get, $y(t) = 3J_0(2t)$

which is the required solution.

Example 7. Solve $[tD^2 + (t-1)D - 1]y = 0$ if $y(0) = 5$, $y(\infty) = 0$.

Solution :

The equation is $ty'' + ty' - y' - y = 0$ ... (1)

with initial conditions $y(0) = 5$ and $y(\infty) = 0$

Taking Laplace transform on both sides of (1) we get

$$L\{ty''\} + L\{ty'\} - L\{y'\} - L\{y\} = 0$$

$$\text{or,}\quad (-1)^1\frac{d}{ds}[L\{y''\}] + (-1)^1\frac{d}{ds}[L\{y'\}] - L\{y'\} - L\{y\} = 0$$

$$\text{or,}\quad -\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] - \frac{d}{ds}[sL\{y\} - y(0)] - [sL\{y\} - y(0)] - L\{y\} = 0 \qquad \cdots (2)$$

Let us put, $L\{y\} = \phi$, and $y'(0) = c$ in (2). Then

$$-\frac{d}{ds}(s^2\phi - 5s - c) - \frac{d}{ds}[s\phi - 5] - [s\phi - 5] - \phi = 0$$

or, $$-\left[s^2\frac{d\phi}{ds} + 2s\phi - 5\right] - \left[s\frac{d\phi}{ds} + \phi\right] - [s\phi - 5] - \phi = 0$$

or, $$(s^2 + s)\frac{d\phi}{ds} + (3s + 2)\phi = 10$$

or, $$\frac{d\phi}{ds} + \frac{3s+2}{s^2+s}\phi = \frac{10}{s^2+s} \quad \text{... (3)}$$

which is a linear differential equation.

Therefore, I.F. $= e^{\int \frac{3s+2}{s^2+s}ds} = e^{\int\left(\frac{2}{s}+\frac{1}{s+1}\right)ds} = s^2(s + 1)$

Multiplying both sides of (3) by $s^2(s + 1)$ and then integrating we get,

$$\phi\, s^2(s + 1) = c_1 + \int \frac{10s^2(s+1)}{s^2+s}ds = c_1 + 10\int s\,ds = c_1 + 5s^2$$

Therefore, $$\phi = \frac{c_1}{s^2(s+1)} + \frac{5}{s+1}$$

or, $$\phi = c_1\left(\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}\right) + \frac{5}{s+1}$$

or, $$L(y) = c_1\left(\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}\right) + \frac{5}{s+1}$$

So, $$y(t) = c_1L^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s} + \frac{1}{s^2}\right\} + 5L^{-1}\left\{\frac{1}{s+1}\right\}$$

or, $$y(t) = c_1(e^{-t} - 1 + t) + 5e^{-t} \quad \text{... (3)}$$

Since it is given that $y(\infty) = 0$, so (3) holds only if $c_1 = 0$

Then from (3) we get

$$y(t) = 5e^{-t}$$

which is the required solution of the equation.

Example 8. Solve $(3D + 2)x + Dy = 1$

$Dx + (4D + 3)y = 0$ **that satisfies the initial condition** $x = 0$, $y = 0$ **when** $t = 0$.

Solution :

The equations are ... (1)

$$3x' + 2x + y' = 1, \quad x' + 4y' + 3y = 0$$

with the conditions $x = 0$, $y = 0$ when $t = 0$.

Taking Laplace transform on both sides of the two given equations in (1),

$$3L\{x'\} + 2L(x) + L\{y'\} = L\{1\}$$

and $$L\{x'\} + 4L(y') + 3L\{y\} = 0$$

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Now writing, $\bar{x}(s) = L\{x\}$ and $\bar{y}(s) = L\{y\}$, we get

$$(3s+2)\bar{x}(s) + s\bar{y}(s) = \frac{1}{s}$$
$$s\bar{x}(s) + (4s+3)\bar{y}(s) = 0$$

[Since $L(x') = sL(x) - x(0)$ and $L(y') = sL(y) - y(0)$]

Solving this system for $\bar{x}(s)$ and $\bar{y}(s)$ we get,

$$\bar{x}(s) = \frac{4s+3}{s(s+1)(11s+6)} = \frac{1}{2s} - \frac{1}{5(s+1)} - \frac{33}{10(11s+6)}$$

and $$\bar{y}(s) = -\frac{1}{(11s+6)(s+1)} = \frac{1}{5}\left(\frac{1}{s+1} - \frac{11}{(11s+6)}\right)$$

Hence $$x(t) = L^{-1}\{\bar{x}(s)\} = L^{-1}\left\{\frac{1}{2s} - \frac{1}{5(s+1)} - \frac{33}{10(11s+6)}\right\}$$

$$= \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-\frac{6}{11}t}$$

and $$y(t) = L^{-1}\{\bar{y}(s)\} = \frac{1}{5}\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{11}{11s+6}\right)\right]$$

$$= \frac{1}{5}\left(e^{-t} - e^{-\frac{6}{11}t}\right)$$

Example 9. Solve $(D^2+2)x - Dy = 1$
$Dx + (D^2+2)y = 0$ given that $x = Dx = y = Dy = 0$ when $t = 0$.

Solution :

The equations are

$$x'' + 2x - y' = 1 \text{ and } x + y'' + 2y = 0 \quad \text{... (1)}$$

with initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Taking the Laplace transform on both sides of the two given equations in (1), we get

$$L\{x''\} + 2L\{x\} - L\{y'\} = L\{1\}$$
and $$L\{x\} + L\{y''\} + 2L\{y\} = 0$$

Now, assuming that $L\{x(t)\} = \bar{x}(s)$ and $L\{y(t)\} = \bar{y}(s)$, we get

$$s^2\bar{x} - sx(0) - x'(0) + 2\bar{x} - \{s\bar{y} - y(0)\} = \frac{1}{s}$$
and $$\bar{x} - s^2\bar{y} - sy(0) - y'(0) + 2\bar{y} = 0$$
i.e., $$s(s^2+2)\bar{x} - s^2\bar{y} - 1 = 0$$
and $$s\bar{x} + (s^2+2)\bar{y} = 0$$ [Since $x'(0) = x(0) = y'(0) = y(0) = 0$]

Solving for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{s^2+2}{s(s^2+1)(s^2+4)} = \frac{1}{2s} - \frac{s}{3(s^2+1)} - \frac{s}{6(s^2+4)}$$

and $$\bar{y} = \frac{-1}{s^4+5s^2+4} = \frac{-1}{(s^2-1)(s^2+4)} = \frac{1}{3(s^2+4)} - \frac{1}{3(s^2+1)}$$

Hence, $x(t) = L^{-1}\{\bar{x}(s)\} = L^{-1}\left\{\frac{1}{2s} - \frac{s}{3(s^2+1)} - \frac{s}{6(s^2+4)}\right\}$

$$= \frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t$$

and $y(t) = L^{-1}\{\bar{y}(s)\} = L^{-1}\left\{\frac{1}{3}\frac{1}{s^2+4} - \frac{1}{3}\cdot\frac{1}{s^2+1}\right\}$

$$= \frac{1}{6}\sin 2t - \frac{1}{3}\sin t$$

Example 10. Solve $Dx + Dy = t$

$D^2x - y = e^{-t}$ if $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$.

Solution :

The equations are

$$x' + y' = t \text{ and } x'' - y = e^{-t} \quad \ldots (1)$$

with the conditions $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$

Taking the Laplace transform on both sides of the two given equations in (1), we get

$$L\{x'\} + L\{y'\} = L\{t\}$$

and $L\{x''\} - L\{y\} = L\{e^{-t}\}$

Let, $\bar{x}(s) = L\{x\}$ and $\bar{y}(s) = L\{y\}$

Then we have

$$s\bar{x} - x(0) + s\bar{y} - y(0)\} = \frac{1}{s^2}$$

and $s^2\bar{x} - sx(0) - x'(0) - \bar{y} = \frac{1}{s+1}$

Using the conditions, we have

$$s\bar{x} + s\bar{y} = 3 + \frac{1}{s^2} = \frac{3s^2+1}{s^2}$$

and $s^2\bar{x} - \bar{y} = 3s - 2 + \frac{1}{s+1} = \frac{3s^2+s-1}{s+1}$

Solving for $\bar{x}$ and $\bar{y}$ we get

$$\bar{x} = \frac{3s^2+1}{s^3(s^2+1)} + \frac{3s^2+s-1}{(s+1)(s^2+1)}$$

$$= \frac{2}{s} + \frac{1}{s^3} + \frac{1}{2(s+1)} - \frac{3}{2(s^2+1)} + \frac{s}{2(s^2+1)}$$

and $\bar{y} = \frac{3s^2+1}{s^2(s^2+1)} - \frac{3s^2+s-1}{(s+1)(s^2+1)}$

$$= \frac{1}{s} - \frac{1}{2(s+1)} + \frac{3}{2(s^2+1)} - \frac{s}{2(s^2+1)}$$

Taking inverse Laplace transforms, we get

$$x = 2 + \frac{1}{2}t^2 + \frac{1}{2}e^{-t} - \frac{3}{2}\sin t + \frac{1}{2}\cos t$$

and $$y = 1 - \frac{1}{2}e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t$$

which is the required solution.

EXERCISE 19(B)

- **Solve :**

1. $(D^2 + 1)y = 0$, if $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$
2. $(D^2 + 3D + 2)y = 0$, Given $y = y_0$ and $Dy = y_1$ at $t = 0$
3. $(D^2 + 4D + 8)y = 0$, when $y(0) = 2$, $y'(0) = 2$
4. $(D^2 + D)x = 2$, when $x(0) = 3$, $x'(0) = 1$
5. $(D + 1)^2 y = t$, given $y = -3$, when $t = 0$ and $y = -1$, when $t = 1$
6. $y''(t) + y(t) = t$, with $y'(0) = 1$, $y(\pi) = 0$
7. $(D^2 + 9)y = 18t$, if $y'(0)$, $y\left(\frac{\pi}{2}\right) = 0$
8. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{2t}$, given $y(0) = -3$ and $y'(0) = 5$
9. $(D^2 - 3D + 2)y = 4t + 3e^t$, given $y = 1$ and $Dy = -1$ when $t = 0$
10. $(D^2 - D - 2)y = 20\sin 2t$, if $y = -1$, $Dy = 2$, when $t = 0$
11. $(D^2 + m^2)x = a\cos nt$, $t > 0$, given $y = Dy = 0$, when $t = 0$
12. $\frac{d^2y}{dt^2} + y = t\cos 2t$, if $y = 0$, $\frac{dy}{dt} = 0$, when $t = 0$
13. $(D^3 + D)y = e^{2t}$, when $y(0) = y'(0) = y''(0) = 0$
14. $(D^3 - D^2 + 4D - 4)y = 68e^t \sin 2t$, given $y = 1$, $Dy = -19$, $D^2y = -37$ at $t = 0$
15. $\{tD^2 + (1 - 2t)D - 2\}y = 0$, given $y(0) = 1$, $y'(0) = 2$
16. $y'' - ty' + y = 1$, if $y(0) = 1$, $y'(0) = 2$
17. $y''(t) + aty'(t) - 2ay(t) = 1$, if $y(0) = y'(0) = 0$, $a > 0$

- **Solve the following simultaneous equations :**

18. $(D - 2)x + 3y = 0$
    $2x + (D - 1)y = 0$, if $x(0) = 8$ and $y(0) = 3$
19. $(D^2 - 3)x - 4y = 0$
    $x + (D^2 + 1)y = 0$, if $x = y = Dy = 0$, $Dx = 2$ and $t = 0$
20. $(D^2 + 3)x = 2y$
    $D^2(x + y) = 3x - 5y$, if $x = y = 0$, $Dx = 3$, $Dy = 2$ at $t = 0$
21. $(D^2 + 2)x - Dy = 1$
    $Dx + (D^2 + 2)y = 0$, if $x = Dx = y = Dy = 0$ when $t = 0$

22. $(D^2 - 1)x + 5Dy = t$
$2Dx - (D^2 - 4)y = 2$, if $x = y = Dx = Dy = 0$ when $t = 0$
23. $2(D + 1)x + (D - 1)y = 3t$
$(D + 1)(x + y) = 1$, if $x = 1, y = 3$ at $t = 0$
24. $Dx + Dy = t$
$D^2x - y = e^{-t}$, if $x(0) = y(0) = x'(0) = 0$
25. $Dx + Dy = t$
$D^2x - y = e^{-t}$, if $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$

Answers

1. $y = \cos t$
2. $y = (2y_0 + y_1)e^{-t} - (y_0 + y_1)e^{-2t}$
3. $y = e^{-2t}(2 \cos 2t + 3 \sin 3t)$
4. $x = e^{-t} + 2t + 2$
5. $y = t - 2 + (t - 1)e^{-t}$
6. $y = t + \pi \cos t$
7. $y = 2t + \pi \sin 3t$
8. $y = 4(t + 1)e^{2t} - 7e^t$
9. $y = 3 + 2t - (3t + 4)e^t + 2e^{2t}$
10. $y = 2e^{2t} - 4e^{-t} + \cos 2t - 3 \sin 2t$
11. $y = \dfrac{a}{m^2 - n^2}(\cos nt - \cos mt)$
12. $y = -\dfrac{5}{9}\sin t + \dfrac{4}{9}\sin 2t - \dfrac{t}{3}\cos 2t$
13. $y = \dfrac{1}{5}(2\cos t - \sin t) + \dfrac{e^{2t}}{10} - \dfrac{1}{2}$
14. $y = \dfrac{1}{5}(e^t + 14 \cos 2t - 3 \sin 2t) - 2e^t(\cos 2t + 4 \sin 2t)$
15. $y = e^{2t}$
16. $y = 1 + 2t$
17. $y = \dfrac{1}{2}t^2$
18. $x = 3e^{4t} + 5e^{-t}$, $y = 5e^{-t} - 2e^{4t}$
19. $x = t(e^t + e^{-t})$, $y = \dfrac{1}{2}\{(1 - t)e^t - (1 + t)e^t\}$
20. $x = \dfrac{11}{4}\sin t + \dfrac{1}{12}\sin 3t$, $y = \dfrac{11}{4}\sin t - \dfrac{1}{4}\sin 3t$
21. $x = \dfrac{1}{2} - \dfrac{1}{3}\cos t - \dfrac{1}{6}\cos 2t$, $y = \dfrac{1}{6}\sin 2t - \dfrac{1}{3}\sin t$
22. $x = 5 \sin t - 2 \sin 2t - t$, $y = \cos 2t - 2\cos t + 1$
23. $x = t + 3e^{-t} - 2e^{-3t}$, $y = 1 - t + 2e^{-3t}$
24. $x = \dfrac{1}{2}(t^2 + e^{-t} + \cos t + \sin t) - 1$, $y = 1 - \dfrac{1}{2}(e^{-t} + \cos t + \sin t)$
25. $x = 2 + \dfrac{1}{2}t^2 + \dfrac{1}{2}e^{-t} - \dfrac{3}{2}\sin t + \dfrac{1}{2}\cos t$, $y = 1 - \dfrac{1}{2}e^{-t} + \dfrac{3}{2}\sin t - \dfrac{1}{2}\cos t$.

19.17 Solution of Partial Differential Equations

Let us assume $\quad L\{y(x, t)\} = \bar{y}(x, s)$,
where $y(x, t)$ is a function of x and t while $\bar{y}(x, s)$ is its Laplace transform of $y(x, t)$ with s as parameter.

i.e., $L\{y(x,t)\} = \int_0^\infty e^{-st}\, y(x,t)dt = \bar{y}(x,s)$,

So, $$L\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial y}{\partial t}dt = \lim_{a\to\infty}\int_0^a e^{-st}\frac{\partial y}{\partial t}dt$$

$$= \lim_{a\to\infty}\left[e^{-st}y(x,t)\Big|_0^a + s\int_0^a e^{-st}y(x,t)dt\right]$$

$$= s\int_0^\infty e^{-st}y(x,t)dt - y(x,0)$$

$$= s\,\bar{y}(x,s) - y(x,0)$$

Again $$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial u}{\partial t}\right\}, \text{ where } u = \frac{\partial y}{\partial t}$$

Hence, $$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial u}{\partial t}\right\} = sL\{u\} - u(x,0)$$

$$= s\{s\,\bar{y}(x,s) - y(x,0)\} - y_t(x,0)$$

$$= s^2\bar{y}(x,s) - sy(x,0) - y_t(x,0)$$

and so on.

Also $$L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial y}{\partial x}dt = \frac{d}{dx}\int_0^\infty e^{-st}y\,dt = \frac{d\bar{y}}{dx}$$

and $$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = L\left\{\frac{\partial v}{\partial x}\right\}, \text{ where } v = \frac{\partial y}{\partial x}$$

$$= \frac{d}{dx}L\{v\} = \frac{d}{dx}L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d}{dx}\left\{\frac{d\bar{y}}{dx}\right\} = \frac{d^2\bar{y}}{dx^2} \text{ and so on.}$$

For some initial and boundary value problems involving partial differential equations, the Laplace transforms provides an effective method for solving these problems. Here we study the application of Laplace transform to find the solutions of partial differential equations with appropriate boundary and initial conditions.

19.18 Application of Laplace Transform to Partial Differential Equations

■ **A Simple problem in heat conduction :** As an example of an application of the transform method to a problem governed by partial differential equation, we consider the classical problem of heat flow in a semi-infinite solid $x > 0$, when the boundary $x = 0$ is kept at a constant temperature V_0, the initial temperature of the solid being zero. If V is the temperature at the time t and ρ is the diffusivity of the material, we have to find V from the partial differential equation

$$\frac{\partial V}{\partial t} = \rho\frac{\partial^2 V}{\partial x^2},\quad x > 0,\ t > 0 \qquad \text{... (1)}$$

with the boundary conditions $V = V_0$ where $x = 0,\ t > 0$ ... (2)

and the initial condition $V = 0$ when $t = 0,\ x > 0$. ... (3)

We write,

$$\overline{V} = \int_0^\infty e^{-st} V dt$$

so that $\overline{V}$ is the Laplace transform of the temperature. Now by integration by parts, ... (4)

$$\int_0^\infty e^{-st}\left(\frac{\partial V}{\partial t}\right)dt = e^{-st}V\Big|_0^\infty + s\int_0^\infty e^{-st}Vdt$$

$$= s\overline{V} \quad \text{using (3) and (4) and taking } s > 0. \quad \text{... (5)}$$

Multiplying (1) by e^{-st} and integrating w.r.t. t from 0 to ∞ we get

$$\int_0^\infty \frac{\partial V}{\partial t} e^{-st} = \rho\int_0^\infty \frac{\partial^2 V}{\partial x^2} e^{-st} dt$$

$$= \rho\int_0^\infty \frac{\partial^2}{\partial x^2}(Ve^{-st})dt$$

$$= \rho\frac{d^2}{dx^2}\int_0^\infty Ve^{-st}dt$$

From (4) and (5), we get

$$\rho\frac{d^2\overline{V}}{dx^2} = s\overline{V}, \quad s > 0 \quad \text{... (6)}$$

with $\overline{V} = \int_0^\infty V_0 e^{-st}dt = \frac{V_0}{s}, \; x = 0$ ... (7)

From the boundary condition, by this procedure, we have reduced the problem to find the solution of the ordinary differential equation (6) with the initial condition (7).

Thus, the general solution of (6) is

$$\overline{V} = Ae^{\sqrt{\frac{s}{\rho}}x} + Be^{-\sqrt{\frac{s}{\rho}}x} \quad \text{... (8)}$$

Since the temparature is finite as $x \to \infty$, we get $A = 0$.

Again applying (7), we get $B = \frac{V_0}{s}$

Then (8) reduces to $\overline{V} = \frac{V_0}{s}e^{-\sqrt{\frac{s}{\rho}}x}$

Taking inverse Laplace transform we get,

$$V = V_0 erfc\left(\frac{x}{2\sqrt{\rho t}}\right) \quad \text{[using (4) in Art. 19.12]}$$

Now some illustrative examples are given to slove partial differential equations with the help of Laplace transforms.

Example 1. Solve $\frac{\partial y}{\partial x} = y + 2\frac{\partial y}{\partial t}$, $y(x, 0) = 6e^{-3x}$, which is bounded for $x > 0$, $t > 0$. [Delhi (Hons.) 1999]

Solution :

Taking Laplace transform of both sides of the given equation, we get

$$L\left\{\frac{\partial y}{\partial x}\right\} = L\{y\} + 2L\left\{\frac{\partial y}{\partial t}\right\}$$

Assuming $L\{y(x, t\} = \bar{y}(x, s)$, we get

$$\frac{d\bar{y}}{dx} = \bar{y} + 2\{s\bar{y} - y(x, 0)\}$$

or, $$\frac{d\bar{y}}{dx} = \bar{y} + 2(s\bar{y} - 6e^{-3x}), \text{ since } y(x, 0) = 6e^{-3x}$$

or, $$\frac{d\bar{y}}{dx} - (2s+1)\bar{y} = -12e^{-3x}$$

which is a linear equation

Therefore, I.F. is $e^{-\int(2s+1)dx} = e^{-(2s+1)x}$

Multiplying both side by I.F. and then integrating, we get

$$\bar{y}\,.\,e^{-(2s+1)x} = c - 12\int e^{-3x}e^{-(2s+1)x}dx$$

$$= c - 12\int e^{-2(s+2)x}dx = c + \frac{6e^{-2(s+2)x}}{s+2}$$

Hence, $$\bar{y}(x, s) = ce^{(2s+1)x} + \frac{6e^{-3x}}{s+2} \quad \text{... (1)}$$

Since $y(x, t)$ is bounded as $x \to \infty$, it follows that $\bar{y}(x, s)$ must be also bounded as $x \to \infty$. This gives $c = 0$.

Therefore (1) reduces to

$$\bar{y}(x, s) = \frac{6e^{-3x}}{s+2} \quad \text{... (2)}$$

Taking inverse Laplace transforms of both sides of (2), we get

$$y(x, t) = 6e^{-3x}L^{-1}\left\{\frac{1}{s+2}\right\} = 6e^{-3x}\,.\,e^{-2t} = 6e^{-(3x+2t)}$$

Hence the required solution is $y(x, t) = 6e^{-(3x+2t)}$.

Example 2. Solve the equation $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = xt$, where $y = 0 = \frac{\partial y}{\partial t}$ at $t = 0$ and $y(0, t) = 0$.

Solution :

The given equation is

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = xt \quad \text{... (1)}$$

where $y(x, 0) = 0$, $y_t(x, 0) = 0$ and $y(0, t) = 0$.

Assuming $L\{y(x, t)\} = \bar{y}(x, s)$ and taking Laplace transformations on both sides of (1) we get

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} - L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = xL\{t\}$$

or, $$\frac{d^2\bar{y}}{dx^2} - [s^2\bar{y}(x, s) - sy(x, 0) - y_t(x, 0)] = \frac{x}{s^2}$$

or, $$\frac{d^2\bar{y}}{dx^2} - s^2\bar{y} = \frac{x}{s^2}$$

or, $$(D^2 - s^2)\,\bar{y} = \frac{x}{s^2} \text{ where } \frac{d^2}{dx^2} \equiv D^2 \quad \text{... (2)}$$

which is a second order linear equation with constant coefficients.

So, C.F. $= c_1e^{sx} + c_2e^{-sx}$, c_1, c_2 being arbitrary constants.

Also P.I. is $$\frac{1}{D^2 - s^2}\left(\frac{x}{s^2}\right) = -\frac{1}{s^4}\left(1 - \frac{D^2}{s^2}\right)^{-1} x$$

$$= -\frac{1}{s^4}\left\{1 + \frac{D^2}{s^2} + \ldots\right\}x = -\frac{x}{s^4}$$

Hence the general solution of (2) is ... (3)

$$\bar{y}(x, s) = c_1e^{sx} + c_2e^{-sx} - \frac{x}{s^4}$$

Now, since $y(x, t)$ is bounded for all x and t, so $\bar{y}(x, s)$ must be bounded. i.e., $\bar{y}(x, s)$ is finite as $x \to \infty$. So we can take $c_1 = 0$ in (3) and hence ... (4)

$$\bar{y}(x, s) = c_2e^{-sx} - \frac{x}{s^4}$$

Here given that $y(0, t) = 0$ ... (5)

Therefore $L\{y(0, t\} = 0$

or, $\bar{y}(0, s) = 0$

Putting $x = 0$ in (4), we get

$\bar{y}(0, s) = c_2e^{-sx} - 0$ i.e. $c_2 = 0$, [using (5)]

and hence from (4)

$$\bar{y}(x, s) = -\frac{x}{s^4}$$

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Therefore, $y(x, t) = L^{-1}\{\bar{y}(x, s)\} = -L^{-1}\left\{\frac{x}{s^4}\right\} = -\frac{xt^3}{3!}$

i.e., $y = -\frac{xt^3}{6}$

which is the required solution.

Example 3. Solve $\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}$, **where $y(0, t) = 0$, $y(2, t)$ and $y(x, 0) = 5 \sin 2\pi x$, $y_t(x, 0) = 0$.**

Solution :

Taking Laplace transform of both sides of the given equation, we get

$$L\left\{\frac{\partial^2 y}{\partial y}\right\} = 9L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

Let $L\{y(x, t)\} = \bar{y}(x, s)$, we get

$$s^2 \bar{y}(x, s) - sy(x, 0) - y_t(x, 0) = 9\frac{d^2\bar{y}}{dx^2}$$

or, $s^2 \bar{y} - 5s \sin 2\pi x = 9\frac{d^2\bar{y}}{dx^2}$

or, $\frac{d^2\bar{y}}{dx^2} - \frac{s^2}{9}\bar{y} = -\frac{5s}{9}\sin 2\pi x$

or, $\left(D^2 - \frac{s^2}{9}\right)\bar{y} = -\frac{5s}{9}\sin 2\pi x \quad \left[D^2 \equiv \frac{d^2}{dx^2}\right]$ ... (1)

which is a linear second order differential equation with constant coefficients.

Putting $\bar{y} = e^{mx}$, A.E. will be $m^2 - \frac{s^2}{9} = 0$, so that $m = \pm\frac{s}{3}$

So, C.F. $= C_1 e^{\frac{s}{3}x} + C_2 e^{-\frac{s}{3}x}$, C_1, C_2 being arbitrary constants.

Now, P.I. $= \frac{1}{D^2 - \frac{s^2}{9}}\left(-\frac{5s}{9}\sin 2\pi x\right) = \frac{1}{-4\pi^2 - \frac{s^2}{9}}\left(-\frac{5s}{9}\sin 2\pi x\right) = \frac{5s \sin 2\pi x}{36\pi^2 + s^2}$

Therefore the general solution of (1) will be

$$\bar{y}(x, s) = C_1 e^{\frac{s}{3}x} + C_2 e^{-\frac{s}{3}x} + \frac{5s \sin 2\pi x}{36\pi^2 + s^2} \quad \text{... (2)}$$

Taking the Laplace transform of the boundary conditions, we have

$$\bar{y}(0, s) = 0 \text{ and } \bar{y}(2, s) = 0 \quad \text{... (3)}$$

Putting $x = 0$ and $x = 2$ in (2) and using the boundary conditions (3), we get

$$C_1 + C_2 = 0 \text{ and } C_1 e^{\frac{2s}{3}} + C_2 e^{-\frac{2s}{3}} = 0$$

Solving for C_1 and C_2, we get, $C_1 = C_2 = 0$

Hence from (2) we get

$$\bar{y}(x, s) = \frac{5s \sin 2\pi x}{s^2 + 36\pi^2} \qquad \text{... (4)}$$

Taking inverse Laplace transform on both sides of (4), we get

$$y(x, t) = 5 \sin 2\pi x L^{-1}\left\{\frac{s}{s^2 + (6\pi)^2}\right\} = 5 \sin 2\pi x \cos 6\pi t$$

Hence the required solution is

$$y(x, t) = 5 \sin 2\pi x \cos 6\pi t.$$

EXERCISE 19(C)

1. Solve $\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}$, where $y(0, t) = y(5, t) = 0$ and $y(x, 0) = 10 \sin 4\pi x$.
2. Solve $\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$, where $y\left(\frac{\pi}{2}, t\right) = 0$; $\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0$ and $y(x, 0) = 30 \cos 5x$.
3. Solve $\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$, $y_x(0, t) = 0$, $y\left(\frac{\pi}{2}, t\right) = 0$ and $y(x, 0) = 20 \cos 3x - 5 \cos 9x$.
4. Find the bounded solution $y(x, t)$, $x > 0$, $t > 0$ of the equation $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$, $y(x, 0) = e^{-3x}$.
5. Solve $\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}$, $y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = -5 \sin 6\pi x$.
6. Find the bounded solution $y(x, t)$, $0 < x < 1$, $t > 0$ of the boundary value problem $\frac{\partial y}{\partial x} - \frac{\partial y}{\partial t} = 1 - e^{-t}$, $y(x, 0) = x$.
7. Solve $3\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ where $y(0, t) = 0$, $y(5, t) = 0$ and $y(x, 0) = -5 \sin 6\pi x$.
8. Solve $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} - 4y$, $y(0, t) = 0$, $y(\pi, t) = 0$; $y(x, 0) = 6 \sin x - 4 \sin 2x$.
9. Find the bounded solution of $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$, $x > 0$, $t > 0$ given that $y(0, t) = 1$ and $y(x, 0) = 0$.

ANSWERS

1. $y(x, t) = 10\, e^{-32\pi^2 t} \sin 4\pi x$
2. $y(x, t) = 30 e^{-75t} \cos 5x$
3. $y(x, t) = 20e^{-27t}\cos 3x - 5e^{-243t} \cos 9x$
4. $y(x, t) = 1 + x - e^{-t}$
5. $y(x, t) = -5e^{-12\pi^2 t} \sin 6\pi x$
6. $y(x, t) = x + 1 - e^{-t}$
7. $y(x, t) = -5e^{-12\pi^2 t} \sin 6\pi x$
8. $y(x, t) = 6e^{-5t}\sin x - 4e^{-8t} \sin 2x$
9. $y(x, t) = erfc\left(\frac{x}{2\sqrt{2}}\right)$.