

Math 2nd Sem (General)

① Superposition Principle Basic theory of Linear diff. Eqn. ② Wronskian Properties.

The linear diff. eqn. of order n with const. Co-efficients is $a_0 \frac{d^2 y}{dx^2} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = R(x)$ ①

① is called non homogeneous diff. Eqn.

If $R(x) = 0$ i.e. $a_0 \frac{d^2 y}{dx^2} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = 0$ is called homogeneous diff. eqn. with const. Coefficient where a_0, a_1, \dots, a_n const.

The symbolic operator D stands for $\frac{d}{dx}$ and

$D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$... The eqn ① Symbolically

may be written as $(a_0 D^n + a_1 D^{n-1} + \dots + a_n) y = R(x)$

and the homogeneous eqn can be written as

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = 0 \dots \dots \dots \textcircled{2}$$

Let y_1, y_2, \dots, y_n be any n solutions of the homogeneous linear diff. eqn. ②

Then $C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ is also a solution of linear diff. eqn. ② where $C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ ③

is called the linear combination of the solutions where C_1, C_2, \dots, C_n arbitrary constants

i.e. If ② D.E. ② has n solutions then each solution multiplied by an arbitrary constant and the all the products then added together. The resulting sum ③ is also a sol. This is the basic theory of linear homogeneous diff. eqn.

The solutions must be linearly independent.

This linearly independent is verified by Wronskian property. If the solutions are ~~not~~ linearly independent, then linear combination of each solution ③ holds good otherwise not. (for general solution)

Existence of linearly independent is verified by
Wronskian. If $y_1(x), y_2(x)$

For 2nd order linear homogeneous diff. eqn³

$$\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \text{ or } (D^2 + P_1 D + P_2)y = 0 \rightarrow (4)$$

Let its linear independent solutions be $y_1(x)$ & $y_2(x)$

then $y = C_1 y_1(x) + C_2 y_2(x)$ or $C_1 y_1 + C_2 y_2$ be also a solution of Eqn³ (4). Now the existence of linearly independent solution is verified by Wronskian.

If $y_1(x), y_2(x)$ be any two solution of (4)

then $W(y_1, y_2)$ or $W(y_1, y_2, x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ Wronskian determinant

ie if $W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$. Then the solutions y_1, y_2 are linearly independent (L. I.) otherwise solutions will be linearly dependent and linear combination will not hold good.

Let the two set³ of any 2nd order diff. homogeneous equation be $y_1 = e^{3x}, y_2 = e^{2x}$, then $y_1' = 3e^{3x}$ and $y_2' = 2e^{2x}$ so $W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{vmatrix}$

$$= 2e^{5x} - 3e^{5x} = -e^{5x} \neq 0$$

hence two solutions are L. I. and linear combination solution $y = C_1 e^{3x} + C_2 e^{2x}$ will be obtained

EX. Show that e^x and xe^x are linearly independent solutions of the D.E. $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 0$. Find the solⁿ if $y(0) = 1, y'(0) = 4$
do yourself

How to find the ^{solution} homogeneous eqn.

$$(D^2 + P_1 D + P_2)y = 0 \Rightarrow \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad \dots (5)$$

Taking $y = e^{mx}$ be a trial sol? y (5)

then it will be found that $Dy = \frac{dy}{dx} = m e^{mx}$
and $D^2 y = \frac{d^2 y}{dx^2} = m^2 e^{mx}$ then putting these values
in Eqn (5) we get

$$m^2 e^{mx} + P_1 m e^{mx} + P_2 e^{mx} = 0$$

or, $e^{mx} (m^2 + P_1 m + P_2) = 0$ but $e^{mx} \neq 0$, as it is sol?

so $m^2 + P_1 m + P_2 = 0$ ~~to find the values of m~~

Eqn (6) is called Auxiliary equation
(A.E) is ~~obtained for~~ from A.E

values of m will be found then C.F = y_c will be
obtained i.e. general solution of ^{homogeneous} diff eqn. (5)

Case (1) If two roots (m_1, m_2) ^{Eqn (6)} be distinct then
C.F or $y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ (where C_1, C_2 arbitrary const.)

Case (2) when two roots of Eqn (6) be equal then C.F
~~or~~ = (Complementary function) $y_c = (C_1 + C_2 x) e^{mx}$

Case (3) If the ~~two~~ roots be complex ^{conjugate} roots i.e. $\alpha \pm i\beta$ then
C.F or $y_c = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$ or $C_1 e^{\alpha x} \cos(\beta x + C_2)$
or $C_1 e^{\alpha x} \sin(\beta x + C_2)$; [C_1, C_2 arbitrary const.]

Case (4) For third order homogeneous diff. Eqn. $(D^3 + P_1 D^2 + P_2 D + P_3)y = 0$
for Complementary Function take $y = e^{mx}$ be a trial sol.

then A.E is $m^3 + P_1 m^2 + P_2 m + P_3 = 0$ from where ~~3~~ values of
 m will be obtained. then C.F or $y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

if 3 values are different of two values equal and third is different
then $y_c = (C_1 + C_2 x) e^{mx} + C_3 e^{m_3 x}$

For detail Consult Diff. eqn by Mukherjee and Brij

Solution of homogeneous (2nd order) Diff. eqnⁿ ^{Note 3} ④ ^{Note 3}

EX ① Solve $3 \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 4y = 0$ or $(3D^2 + 8D + 4)y = 0$ — ①

Solⁿ: Let $y = e^{mx}$ be a trial solⁿ.

Then $Dy = me^{mx}$, $D^2y = m^2e^{mx}$ putting the values in ① we get $e^{mx}(3m^2 + 8m + 4) = 0$ but $e^{mx} \neq 0$.

So $3m^2 + 8m + 4 = 0 \Rightarrow (m+2)(3m+2) = 0$

or, $m = -2, -\frac{2}{3}$ ~~different~~ ^{different} value.

\therefore Solution or C.F. or $y_c = C_1 e^{-2x} + C_2 e^{-\frac{2}{3}x}$

EX ② Solve $(D^2 + 8D + 16)y = 0$

Solⁿ: Let $y = e^{mx}$ be a trial solⁿ. Then A.E. is

given by $m^2 + 8m + 16 = 0$ or, $(m+4)^2 = 0 \Rightarrow m = -4, -4$

So the solution will be $y = (A + Bx)e^{-4x}$, A, B arbitrary Consts.

EX ③ Solve $\frac{d^2 x}{dt^2} - 4 \frac{dx}{dt} + 13x = 0$ — ①

Solⁿ: Let $x = e^{mt}$ be a trial solⁿ. Then $Dx = me^{mt}$

$D^2x = m^2e^{mt}$ where $D = \frac{d}{dt}$

Putting the values in ①

we get $e^{mt}(m^2 - 4m + 13) = 0 \Rightarrow e^{mt} \neq 0$ so $m^2 - 4m + 13 = 0$

$\therefore m = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$

So the solⁿ will be

EX ④ Solve $\frac{d^2 x}{dt^2} + n^2 x = 0$ — ① $y = e^{2t}(C_1 \cos 3t + C_2 \sin 3t)$

Solⁿ: Let $x = e^{mt}$ be a solⁿ of the eqnⁿ. Then finding & putting

values of $\frac{d^2 x}{dt^2}$ and $\frac{dx}{dt}$, x in ① we get A.E. $e^{mt}(m^2 + n^2) = 0$

$e^{mt} \neq 0$ so $m^2 + n^2 = 0$ $\therefore m = -n = ni$ $\therefore m = 0 \pm ni = \pm ni$

So the general solⁿ is $x = (A \cos nt + B \sin nt)e^{0t}$

or $x = (A \cos nt + B \sin nt)$ where A, B arbitrary Consts.