Semester - VI

Course Type - DSE-4

Course Title: DSE-4: Mathematics Modelling

Topic: Bessel Equation

References: M.D. Raisinghania Book

Date: 01.05.2020

— Shambhu Nath Acharya

11

Bessel Functions

11.1. Bessel's equation and its solution. **[Garhwal 2004; Kanpur 2009]**

The differential equation of the form $$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad ...(1)$$

or $$y'' + (1/x) \times y' + (1 - n^2/x^2)y = 0 \qquad ...(1)'$$

is called *Bessel's equation of order n*, n being a non-negative constant. We now solve (1) in series by using the well known method of Frobenius.

Let the series solution (1) be $$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0. \qquad ...(2)$$

$$\therefore \; y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}.$$

Substitution for y, y', y'' in (1) now gives

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + x \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + (x^2 - n^2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) + (k+m) - n^2\} x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0.$$

But the bracketed expression in the above identity

$$= (k+m)^2 - (k+m) + (k+m) - n^2 = (k+m)^2 - n^2 = (k+m+n)(k+m-n).$$

So the above identity becomes

$$\sum_{m=0}^{\infty} c_m (k+m+n)(k+m-n) x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} \equiv 0. \qquad ...(3)$$

Equating to zero the smallest power of x, namely x^k, (3) gives the indicial equation

$c_0(k+n)(k-n) = 0$ *i.e.*, $(k+n)(k-n) = 0$ as $c_0 \neq 0$. Its roots are $k = n, -n$.

Next equating to zero the coefficient of x^{k+1} in (3) gives

$$c_1(k+1+n)(k+1-n) = 0, \quad \text{so that} \quad c_1 = 0 \quad \text{for} \quad k = n \quad \text{and} \quad k = -n.$$

Finally equating to zero the coefficient of x^{k+m} in (3) gives

$$c_m(k+m+n)(k+m-n) + c_{m-2} = 0 \quad \text{or} \quad c_m = \frac{1}{(k+m+n)(n-k-m)} c_{m-2}. \qquad ...(4)$$

Putting $m = 3, 5, 7, ...$ in (4) and using $c_1 = 0$, we find

$$c_1 = c_3 = c_5 = c_7 = ... = 0. \qquad ...(5)$$

Putting $m = 2, 4, 6, ...$ in (4) gives $$c_2 = \frac{1}{(k+2+n)(n-k-2)} c_0,$$

$$c_4 = \frac{1}{(k+4+n)(n-k-4)} c_2 = \frac{1}{(k+4+n)(n-k-4)(k+2+n)(n-k-2)} c_0$$

and so on. Putting these values in (2), we get

$$y = c_0 x^k \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(n+k+4)(n-k-4)} + \ldots\right]$$

Replacing k by n and $-n$ and also replacing c_0 by a and b in the above equation gives

$$y = ax^n \left\{1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8(1+n)(2+n)} - \ldots\right\} \qquad \ldots(6)$$

and

$$y = bx^{-n} \left\{1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8(1-n)(2-n)} - \ldots\right\}. \qquad \ldots(7)$$

The particular solution of (1) obtained from (6) above by taking the arbitrary constant $a = 1/\{2^n\,\Gamma(n+1)\}$, is called the *Bessel function of the first kind of order n*. It will be denoted by $J_n(x)$. Thus, we have

$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8(n+1)(n+2)} - \ldots\right] \qquad \ldots(8)$$

or

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \qquad \ldots(9)$$

Replacing b by $1/\{2^n\,\Gamma(n+1)\}$ in (7) and proceeding as above gives

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n} \qquad \ldots(10)$$

Let n be non–integral. We know that $\Gamma(m) = \infty$ if m is zero or a negative integer and $\Gamma(m)$ is finite otherwise. Since n is not an integer and r is always integral, the factor $\Gamma(-n+r+1)$ in (10) is always finite and non–zero. For $2r < n$, (10) shows that $J_{-n}(x)$ contains negative powers of x. On the other hand, (9) shows that $J_n(x)$ is not containing negative powers of x at all. Therefore, we find that at $x = 0$, $J_n(x)$ is finite while $J_{-n}(x)$ is infinite, and so one cannot be expressed as a constant multiple of the other. From these arguments we conclude that $J_n(x)$ and $J_{-n}(x)$ are two independent solutions of (1) when n is not an integer (this condition being stronger than $2n$ non–integral which was assumed earlier). Thus, the general solution of Bessel equation (1) when n is not an integer is

$y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constant. ... (11)

11.2. Bessel's functions of the first kind of order n. Definition [Nagpur 2003]

Bessel's function of the first kind and of order n is denoted by $J_n(x)$ and is defined as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}, \text{ where } n \text{ is any non–negative constant.} \qquad \ldots(1)$$

Remark 1. When n is an integer, $\Gamma(n+r+1) = (n+r)!$ and so (1) may be rewritten as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{2r+n}. \qquad \ldots(2)$$

Replacing n by 0 and 1 in turn in (2), Bessel's functions of orders 0 and 1 are given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \qquad \text{Kanpur 2005]}$$

and $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 3} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \ldots$$

Remark 2. When there is no confusion regarding the variable, we shall write J_n for $J_n(x)$ and J'_n for $d\,J_n(x)/dx$ etc.

11.3. List of important results of Gamma function Γ(n) and Beta function B(m, n).

For more details, refer Chapter 6.

(i) $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx,\ n > 0$ (ii) $\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$

(iii) $\Gamma(1) = 1$ (iv) $\Gamma(1/2) = \sqrt{\pi}$

(v) $\Gamma(n+1) = n\,\Gamma(n),\ n > 0$ (vi) $\Gamma(n+1) = n\,!$, if n is +ve integer

(vii) $B(m, n) = B(n, m)$ (viii) $\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi$.

(ix) $\Gamma(m) = \infty$, so that $1/\Gamma(m) = 0$ if $m = 0$ or –ve integer

(x) $B(m, n) = \int_0^i x^{m-1}(1-x)^{n-1}\, dx = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$, where $m > 0,\ n > 0$

(xi) $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ (xii) $\Gamma(2n) = \dfrac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\,\Gamma(n+\tfrac{1}{2})$

11.4. Relation between $J_n(x)$ and $J_{-n}(x)$, n being an integer

Theorem. I Show that when n is

(i) positive integer, $J_{-n}(x) = (-1)^n J_n(x)$. **[Agra 2006; Kanpur 2006, 07, 08 MDU Rohtak 2004; Purvanchal 2006; Meerut 2006; Nagpur 1995;]**

(ii) any integer, $J_{-n}(x) = (-1)^n J_n(x)$ **[Kanpur 2004, 08; Ranchi 2010; Meerut 1993]**

Proof. Part (i). Let n be a +ve integer. We know that

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}. \qquad ...(1)$$

Since $n > 0$, so $\Gamma(-n+r+1)$ is infinite (and so $1/\Gamma(-n+r+1)$ is zero) for $r = 0, 1, 2, \ldots, (n-1)$. Keeping this in mind we see that the sum over r in (1) must be taken from n to infinity. Thus,

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \qquad ...(2)$$

From (2), $J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^{m+n} \dfrac{1}{(m+n)\,!\,\Gamma(m+1)} \left(\dfrac{x}{2}\right)^{2(m+n)-n}$, (on changing the variable of summation to $m = r - n$ so that $r = m + n$ and so $m = 0$ when $r = n$ and $m = \infty$ when $r = \infty$)

$$\therefore \quad J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m (-1)^n \frac{1}{\Gamma(m+n+1)\,m!}\left(\frac{x}{2}\right)^{2m+n} = (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(r+n+1)\,r!}\left(\frac{x}{2}\right)^{2m+n},$$

(on changing the variable of summation from m to r while keeping the limits of summation unchanged.)

Thus, for $n > 0$ $\quad J_{-n}(x) = (-1)^n J_n(x)$, by the definition of $J_n(x)$. ... (3)

Part (ii). Let n < 0. Let p be a positive integer such that $n = -p$. Since $p > 0$, from part (i) above, we have $J_{-p}(x) = (-1)^p J_p(x)$ so that $J_p(x) = (-1)^{-p} J_{-p}(x)$.

But $p = -n$ hence the above result becomes $J_{-n}(x) = (-1)^n J_n(x)$, ...(4)

which is of the same form as (3). Hence the required result holds for any integer.

Remark. When n is an integer $J_{-n}(x)$ is not independent of $J_n(x)$, because $J_{-n}(x)$ is a constant multiple of $J_n(x)$ as shown above. Hence $y = AJ_n(x) + BJ_{-n}(x)$ is not the general solution of Bessel equation when n is an integer. Of course, when n is not an integer, the most general solution of Bessel equation is given by $y = AJ_n(x) + BJ_{-n}(x)$. When n is an integer, the nature of general solution is indicated by the following theorem.

Theorem II. The two independent solutions of Bessel's equation may be taken to be $J_n(x)$ and

$$Y_n(x) = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi}, \text{ for all values of } n. \qquad ...(5)$$

Proof. Case I. Let n be not an integer. Since n is not an integer, $\sin n\pi \neq 0$. Hence (5) shows that $Y_n(x)$ is a linear combination of $J_n(x)$ and $J_{-n}(x)$. But we know that $J_n(x)$ and $J_{-n}(x)$ are independent solutions if n is not an integer. Hence $J_n(x)$ and a linear combination of $J_n(x)$ and $J_{-n}(x)$ will also be independent solutions. Thus we find that $J_n(x)$ and $Y_n(x)$ are two independent solutions of Bessel's equation.

Case II. Let n be an integer. Then we have $\cos n\pi = (-1)^n$, $\sin n\pi = 0$ and $J_{-n}(x) = (-1)^n J_n(x)$.

Using these values in (5), we find that $Y_n(x)$ has the form 0/0 and so $Y_n(x)$ is undefined. To make $Y_n(x)$ meaningful, we define it as

$$Y_n(x) = \lim_{v \to n} Y_v(x) = \lim_{v \to n} \frac{\cos v\pi\, J_v(x) - J_{-v}(x)}{\sin v\pi} \qquad ...(6)$$

$$= \frac{[(\partial/\partial v)\{(\cos v\pi\, J_v(x) - J_{-v}(x)\}]_{v=n}}{[(\partial/\partial v)\cos v\pi)]_{v=n}}, \text{ by L' Hospital's rule}$$

$$= \frac{\left[-\pi \sin v\pi\, J_v(x) + \cos v\pi \frac{\partial}{\partial v} J_v(x) - \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{[\pi \cos v\pi]_{v=n}} = \frac{\cos n\pi \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi \cos n\pi}$$

$$= \frac{(-1)^n \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - (-1)^{2n} \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi (-1)^n} = \frac{1}{\pi}\left[\frac{\partial}{\partial v} J_v(x) - (-1)^n \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n} \qquad ...(7)$$

We now establish the following two results about $Y_n(x)$ as given by (6).

(*i*) $Y_n(x)$ is a solution of Bessel's equation. (*ii*) $Y_n(x)$ is a solution independent of $J_n(x)$.

Proof of (i). Since $J_v(x)$ and $J_{-v}(x)$ are solutions of Bessel's equation of order v, we must have

$$x^2 \frac{d^2 J_v}{dx^2} + x \frac{dJ_v}{dx} + (x^2 - v^2) J_v = 0 \qquad ...(8)$$

and

$$x^2 \frac{d^2 J_{-v}}{dx^2} + x \frac{dJ_{-v}}{dx} + (x^2 - v^2) J_{-v} = 0. \qquad ...(9)$$

Differentiating (8) and (9) w.r.t. 'v', we obtain

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right) + x \frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right) + (x^2 - v^2) \frac{\partial J_v}{\partial v} - 2vJ_v = 0 \qquad ...(10)$$

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right) + x \frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right) + (x^2 - v^2) \frac{\partial J_{-v}}{\partial v} - 2vJ_{-v} = 0. \qquad ...(11)$$

Multiplying (11) by $(-1)^v$ and subtracting from (10) gives

$$x^2 \frac{d^2}{dx^2}\left\{\frac{\partial J_v}{\partial v} - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\} + x \frac{d}{dx}\left\{\frac{\partial}{\partial v} J_v - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\} + (x^2 - v^2)\left\{\frac{\partial}{\partial v} J_v - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\}$$

$$- 2v\{J_v - (-1)^v J_{-v}\} = 0 \qquad ...(12)$$

$$x\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2}+\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1}+x\sum_{m=0}^{\infty}c_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-1}+\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1}+\sum_{m=0}^{\infty}c_m x^{k+m+1}=0$$

or $$\sum_{m=0}^{\infty}\{(k+m)(k+m-1)+(k+m)\}c_m x^{k+m-1}+\sum_{m=0}^{\infty}c_m x^{k+m+1}=0$$

or $$\sum_{m=0}^{\infty}(k+m)^2 c_m x^{k+m-1}+\sum_{m=0}^{\infty}c_m x^{k+m+1}=0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the lowest power of x, namely x^{k-1}, we have

$k^2 c_0 = 0$ so that $k = 0, 0$ (as $c_0 \neq 0$)

Now equating to zero the coefficient of next higher power of x, namely x^k, in (4), we have

$c_1(k+1)^2 = 0$ so that $c_1 = 0$ as $(k+1)^2 \neq 0$ for $k = 0$.

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k+m)^2 c_m + c_{m-2} = 0 \qquad \text{or} \qquad c_m = -\frac{c_{m-2}}{(k+m)^2}, \text{ for all } m \geq 2.$$

When $k = 0$, we have $\qquad c_m = -(1/m^2)c_{m-2}$ for all $m \geq 2 \qquad ...(5)$

Putting $m = 3, 5, 7, ...$ in (5) and noting that $c_1 = 0$, we obtain

$$c_1 = c_3 = c_5 = c_7 = ... = 0. \qquad ...(6)$$

Next, putting = 2, 4, 6, ... in (5), we have

$$c_2 = -\frac{c_0}{2^2}, \qquad c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2 \cdot 4^2}, \qquad c_6 = -\frac{c_4}{6^2} = -\frac{c_0}{2^2 \cdot 4^2 \cdot 6^2}, \text{ ... and so on}$$

Putting the above values in (2) for $k = 0$, $\quad y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$

or $$y = c_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \text{......... ad. inf}\right)$$

If $c_0 = 1$, the above solution is denoted by $J_0(x)$ so that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \text{......... ad. inf}, \qquad ...(7)$$

where $J_0(x)$ is known as *Bessel's function of zeroeth order.*

Note 1. Replacing n by 0 in Art. 11.2, we can deduce (7).

Note 2. From (7), we have $\qquad J_0(0) = 1. \qquad$ **[Nagpur 1995]**

11.6A. Solved examples based on Art. 11.1 to 11.6

Ex. 1. *Prove that*

(*i*) $J_{-1/2}(x) = \sqrt{(2/\pi x)}\cos x.$ **[Garhwal 2005; Nagpur 2005; Kanpur 2009, 10; Agara 2010; Bhopal 2010; Ranchi 2010]**

(*ii*) $J_{1/2}(x) = \sqrt{(2/\pi x)}\sin x.$ **[Nagpur 2003, 05; Garhwal 2004; Kanpur 2004; 07]**

(*iii*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = 2/(\pi x).$ **[Lucknow 2010; Meerut 1992, 93]**

Sol. By the definition of $J_n(x)$, we have

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2 \cdot 2(n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2(n+1)(n+2)} - ...\right] \qquad ...(1)$$

Part (i). Replacing n by – (1/2) in (1) and simplifying, we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma\left(\frac{1}{2}\right)}\left[1-\frac{x^2}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-...\right] = \sqrt{\frac{2}{\pi x}}\cos x, \text{ as } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Part (ii). Replacing n by 1/2 in (1) and simplifying, we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)}\cdot\left[1-\frac{x^2}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}-...\right] = \sqrt{\left(\frac{x}{2}\right)}\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\cdot\frac{1}{x}\left[x-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-...\right]$$

$[\because \Gamma(p+1) = p\Gamma(p)]$

$$= \sqrt{\frac{2}{\pi x}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

Part (iii). Squaring and adding the results of (*i*) and (*ii*), we get

$$[J_{1/2}(x)]^2 + J_{-1/2}(x)]^2 = (2/\pi x)(\sin^2 x + \cos^2 x) = 2/\pi x.$$

Ex. 2. *Prove that* $\lim_{z\to 0}\frac{J_n(z)}{z^n} = \frac{1}{2^n\Gamma(n+1)^n}$, *where* $n > -1$. **(Kanpur 2005, 07)**

Sol. By defination, $J_n(z) = \frac{z^n}{2^n\,\Gamma(n+1)}\left[1-\frac{z^2}{4(n+1)}+\frac{z^4}{4\,.\,8\,.\,(n+1)\,.\,(n+2)}-...\right]$

$$\therefore \quad \lim_{z\to 0}\frac{J_n(z)}{z^n} = \frac{1}{2^n\,\Gamma(n+1)}\left[1-\frac{z^2}{4(n+1)}+\frac{z^4}{4\,.\,8\,.\,(n+1)\,.\,(n+2)}-...\right] = \frac{1}{2^n\Gamma(n+1)}$$

Ex. 3. *Write the general solution of the following equations:*

(*i*) $x^2(d^2y/dx^2) + x(dy/dx) + (x^2 - 25)y = 0$ **(MDU Rohtak 2005)**

(*ii*) $x^2(d^2y/dx^2) + x(dy/dx) + (x^2 - 9/16)y = 0$

(*iii*) $d^2y/dx^2 + (1/x) \times (dy/dx) + (1 - 1/6.25x^2)\,y = 0$

(*iv*) $x^2\,(d^2z/dx^2) + x(dz/dx) + (x^2 - 64)\,z = 0$

(*v*) $z(d^2y/dz^2) + (dy/dz) + zy = 0$

Sol. In what follows, we shall use the following solutions of Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Result I: $y = A\,J_n\,(x) + B\,J_{-n}(x)$, where n is not an integer. A, B being arbitrary constants.

Result II: $y = A\,J_n(x) + B\,Y_n(x)$, where n is an integer, A, B being arbitrary constants.

(*i*) Given $x^2y'' + xy' + (x^2 - 5^2)y = 0$, which is Bessel's equation of order 5, which is an integer. Its general solution is $y = A\,J_5(x) + B\,Y_5(x)$, where A and B are arbitrary constants.

(*ii*) Given $x^2y'' + xy' + \{x^2 - (3/4)^2\} = 0$, which is Bessel's equation of order 3/4, which is not an integer. Its solution is $y = A\,J_{3/4}(x) + B\,J_{-3/4}\,(x)$, where A and B are arbitrary constants

(*iii*) Re-writing, given equating becomes $x^2y'' + xy' + \{x^2 - (2/5)^2\}y = 0$

As in part (*ii*), solution is $y = A\,J_{2/5}(x) + B\,J_{-2/5}(x)$, A, B being arbitrary constants

(*iv*) Given $x^2(d^2z/dx^2) + x(dz/dx) + (x^2 - 8^2)\,z = 0$

As in part (*i*), solution is $z = A\,J_8\,(x) + B\,Y_8(x)$, A, B being arbitrary constants

(*v*) Re-writing the given equation, $z^2(d^2y/dz^2) + z(dy/dz) + z^2y = 0$

which is a Bessel equation of order 0, which is an integer. Its solution is $y = A\,J_0(z) + BY_0(z)$.

Using value of $J_n(x)$ and the above expression, (13) gives

$$\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2\cdot 2(n+1)}+\ldots\right]=kx^n\left[\frac{(-1)^n n!}{(2^n n!)^2}+\text{terms contaning } x^2\right]$$

Equaing the coefficients of x^n from both sides, we get

$$\frac{1}{2^n\Gamma(n+1)}=k\frac{(-1)^n\, n!}{(2^n n!)^2} \quad \text{or} \quad k=(-1)^n\,2^n=(-2)^n \qquad \ldots (15)$$

[$\because$ $\Gamma(n+1)=n\,!$, n being +ve integer]

With this value of k, (13) gives $\qquad J_n(x)=(-2)^n x^n \dfrac{d^n J_0(x)}{d(x^2)^n}$.

(*b*) Proceed as in part (*a*). From (15), we have $k=(-2)^n$

EXERCISE 11 (A)

1. Show that $y=x^{-n/2}\,J_n(2\sqrt{x})$ satisfies the equation $xy''+(n+1)y'+y=0$.

2. Show that $\int_0^{\pi/2}\sqrt{(\pi x)}\,J_{1/2}(2x)\,dx=1$.

3. (*a*) Prove : $\int_0^\infty J_n(bx)x^n e^{-ax}dx=\dfrac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\cdot\dfrac{b^n}{(a^2+b^2)^{n+1/2}}$

(*b*) Prove that $\int_0^\infty J_n(bx)\,x^{n+1}\,e^{-ax}dx=\dfrac{2^{n+1}\Gamma(n+\frac{1}{2})}{\pi}\cdot\dfrac{ab^n}{(a^2+b)^{n+3/2}}$, $a>0$.

4. Prove : $\int_0^\infty J_n(bx)\,x^{n+1}e^{-ax^2}dx=\dfrac{b^n}{(2a)^{n+1}}\exp(-b^2/4a)$, where $\exp(p)=e^p$

5. Prove that $\int_0^\infty J_n(bx)x^{n+1}e^{-ax^2}dx=\dfrac{b^n}{2^{n+1}a^{n+2}}\left(n+1-\dfrac{b^2}{4a}\right)\exp\left(-\dfrac{b^2}{4a}\right), a>0$.

6. For what value of n the general solution of Bessel's differential equation will be of the form $y=AJ_n(x)+BJ_n(x)$.

7. Write the differential equation satisfied by Bessel's function of order n. Express the following Bessel's functions in terms of trigonometric functions :

(*i*) $J_{1/2}(x)$, (*ii*) $J_{-1/2}(x)$, (*iii*) $J_{3/2}(x)$, (*iv*) $J_{-3/2}(x)$.

11.7 Recurrence Relations (Formulae) for $J_n(x)$. Prove that

I. $\dfrac{d}{dx}\{x^n J_n(x)\}=x^n J_{n-1}(x)$. **[Agra 2005; Guwahati 2007; Kanpur 2004, 09; Nagpur 1996]**

II. $\dfrac{d}{dx}\{x^{-n}J_n(x)\}=-x^{-n}J_{n+1}(x)$. **[Agra 2008; Gulbarga 2005; Kanpur 2009]**

III. $J'_n(x)=J_{n-1}(x)-(n/x)\,J_n$ or $xJ'_n=-nJ_n+xJ_{n-1}$. **[Agra 2010; Bilaspur 2004; KU Kurukshetra 2005; Agra 1997; Kanpur 2005, 09, 11; Meerut 2010; Nagpur 2010]**

IV. $J'_n(x)=(n/x)\,J_n(x)-J_{n+1}(x)$ or $xJ'_n=nJ_n-xJ_{n+1}$. **[Nagpur 2005; Bangalore 93, 94; Meerut 2005, 07, 11; Kanpur 1999; Bilaspur 1998]**

V. $J'_n(x)=\frac{1}{2}\{J_{n-1}(x)-J_{n+1}(x)\}$ or $J_{n-1}-J_{n+1}=2J'_n$.
Agra 2009; Jiwaji 2004, 07; Ravishankar 2004; Nagpur 1995; Kanpur 2007]

VI. $J_{n-1}(x)+J_{n+1}(x)=(2n/x)J_n(x)$ or $xJ_{n+1}(x)+xJ_{n-1}(x)=2nJ_n(x)$
or $2J_n=x(J_{n-1}+J_{n+1})$. **[Agra 2008; Bangalore 2005; Meerut 2006; Kanpur 2006, 11; Nagpur 2010]**

Proof 1. Using the definition of $J_n(x)$, we have

$$\frac{d}{dx}\{x^n J_n(x)\} = \frac{d}{dx}\left\{x^n \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n}} \cdot \frac{d}{dx}(x^{2r+2n}) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(2r+2n)\,x^{2r+2n-1}}{r!\Gamma(n+r+1)\,2^{2r+n}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2 \cdot (r+n)}{r!\,(n+r)\,\Gamma(n+r)} \cdot \frac{x^n \cdot x^{2r+n-1}}{2^{2r+n}} \qquad [\because\ \Gamma(n+1) = n\Gamma(n)]$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)}\left(\frac{x}{2}\right)^{2r+n-1} = x^n J_{n-1}(x). \text{ by the definition of } J_{n-1}(x).$$

II. Using the definition of $J_n(x)$, we have

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = \frac{d}{dx}\left\{x^{-n} \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n}} \cdot \frac{d}{dx}(x^{2r}) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2rx^{2r-1}}{r(r-1)!\,\Gamma(n+r+1)\,2^{2r+n}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{2r-1}}{(r-1)!\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n-1}} = \sum_{r=1}^{\infty} \frac{(-1)^r\, x^{2r-1}}{(r-1)!\Gamma(n+r+1)} \cdot \frac{1}{2^{2r+n-1}}$$

(since $(r-1)! = \infty$ when $r = 0$ so the term corresponding to $r = 0$ vanishes)

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2m+2-1}}{m!\Gamma(n+m+2)} \cdot \frac{x^n \cdot x^{-n}}{2^{2m+2+n-1}}, \text{ (on changing the variable of summation to } m = r-1$$

so that $r = m + 1$. Then $m = 0$ when $r = 1$ and $m = \infty$ when $r = \infty$)

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+2)}\left(\frac{x}{2}\right)^{n+1+2m} = -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+1+r+1)}\left(\frac{x}{2}\right)^{n+1+2r}$$

(on changing the variable of summation from m to r)

$= -x^{-n} J_{n+1}(x)$, by the definition of $J_{n+1}(x)$.

III. Recurrence relation I is

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x) \quad \text{or} \quad n\,x^{n-1} J_n(x) + x^n J'_n(x) = x^n J_{n-1}(x).$$

Dividing both sides by x^{n-1}, $\quad n\,J_n(x) + x\,J'_n(x) = x\,J_{n-1}(x)$

or $\quad (n/x)\,J_n(x) + J_n'(x) = J_{n-1}(x) \quad$ or $\quad J'_n(x) = J_{n-1}(x) - (n/x)\,J_n(x)$

IV. Recurrence relation II is

$$\frac{d}{dx}\{x^{-n}\,J_n(x)\} = -x^{-n}\,J_{n+1}(x) \quad \text{or} \quad -n\,x^{-n-1} J_n(x) + x^{-n}\,J'_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing both sides by x^{-n}, $\quad n\,x^{-1}\,J_n(x) + J'_n(x) = -J_{n+1}(x)$

or $\quad (-n/x)\,J_n + J'_n = J_{n+1} \quad$ or $\quad J'_n = (n/x)\,J_n - J_{n+1}$.

V. From recurrence relations III and IV, we have $J'_n(x) = J_{n-1}(x) - (n/x)J_n(x)$ ...(1)

and $J'_n(x) = (n/x)\, J_n(x) - J_{n+1}(x)$. ...(2)

Adding (1) and (2), $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ or $J'_n(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$.

VI. From recurrence relations III and IV, we have $J'_n(x) = J_{n-1}(x) - (n/x)J_n(x)$ ...(1)

and $J'_n(x) = (n/x)\, J_n(x) - J_{n+1}(x)$. ...(2)

Subtracting (2) from (1), we get

$0 = J_{n-1} + J_{n+1} - 2(n/x)\, J_n$ or $J_{n-1} + J_{n+1} = (2n/x)\, J_n$.

11.7. A. Solved examples based on recurrence relations

Ex. 1(*a*). *Show that* $x^n J_n(x)$ is a solution of $x(d^2y/dx^2) + (1-2n) \times (dy/dx) + xy = 0$.

[CDLU 2004]

(*b*) *Show that* $x^{-n}J_n(x)$ is a solution of $x(d^2y/dx^2) + (1+2n) \times (dy/dx) + xy = 0$.

[MDU Rohtak 2005]

Sol. (*a*) Given $x(d^2y/dx^2) + (1-2n) \times (dy/dx) + xy = 0$ ... (1)

Let $y = x^n J_n(x)$ ... (2)

From recurrence relation I, $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

or $dy/dx = x^n J_{n-1}$, using (2) ... (3)

From (3), $d^2y/dx^2 = x^n J'_{n-1} + n\, x^{n-1} J_{n-1}$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x(x^n J'_{n-1} + nx^{n-1}J_{n-1}) + (1-2n)x^n J_{n-1} + x^{n+1}J_n = 0 \quad \text{or} \quad x^{n+1}J'_{n-1} - (n-1)x^n J_{n-1} + x^{n+1}J_n = 0$$

or $$x^{n+1}\left[J'_{n-1} - \{(n-1)/x\}\, J_{n-1}\right] + x^{n+1}J_n = 0 \qquad \text{... (4)}$$

From recurrence relation VI, we have

$$J'_n(x) = \frac{n}{x}J_n - J_{n+1}(x) \quad \text{so that} \quad J'_{n-1} - \frac{n-1}{x}J_{n-1} = -J_n(x) \qquad \text{... (5)}$$

Using (5), (4) reduces to $-x^{n+1} J_n + x^{n+1} J_n = 0$, *i.e.*, $0 = 0$, which is true.

Hence $x^n J_n$ is a solution of (1).

(*b*) Given $x(d^2y/dx^2) + (1+2n) \times (dy/dx) + xy = 0$ ... (1)

Let $y = x^{-n}J_n(x)$ ... (2)

From recurrence relation II, $\dfrac{d}{dx}\left\{x^{-n}J_n(x)\right\} = -x^{-n}J_{n+1}(x)$

or $dy/dx = -x^{-n}J_{n+1}$, using (2)

From (3), $d^2y/dx^2 = -x^{-n}J'_{n+1} + n\, x^{-n-1}J_{n+1}$ ... (3)

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x(-x^{-n}J'_{n+1} + nx^{-n-1}J_{n+1}) + (1+2n)\times(-x^{-n}J_{n+1}) + x^{-n+1}J_n = 0$$

or $-x^{-n+1}J'_{n+1} - (n+1)x^{-n}J_{n+1} + x^{-n+1}J_n = 0$ or $-x^{-n+1}\left[J'_{n+1} + \{(n+1)/x\}\, J_{n+1}\right] + x^{-n+1}J_n = 0$ ... (4)

From recurrence relation III, we have

$$J'_n(x) = J_{n-1}(x) - \frac{n}{x}J_n \quad \text{so that} \quad J'_{n+1} + \frac{n+1}{x}J_{n+1} = J_n(x) \qquad \text{... (5)}$$