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Math

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for Sem - 6 (H) students: (Complex Analysis)
Paper - C13T.

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• Mappings

Consult Book: Complex Analysis by Alan Jeffrey.

An important part of the study of analytic function is concerned with their geometrical properties. The geometrical properties of analytic functions form a subject called conformal mapping.

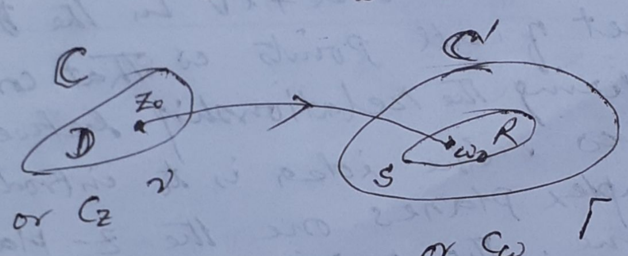
The complex function $w = f(z) = u(x,y) + iv(x,y)$ for $z \in D$ relates the points $z = x+iy$ in Domain of definition D of f to points ~~a~~ $w = u+iv$ in the range R of f i.e. R is the set of all points w that corresponds to points z in D. Considering the relationship between complex variables z and w. The idea is to introduce two different ~~complex~~ complex planes one the z-plane and the other the w-plane. The points in the w-plane determined by $w = f(z)$ for $z \in D$. This is due to Riemann.

Let us denote by $\mathbb{C}$ or $\mathbb{C}_z$, the set of points comprising the entire z-plane and be $\mathbb{C}'$ or $\mathbb{C}_w$ the set of points comprising the entire ~~pla~~ w-plane. If the function f has the domain of definition D, the correspondence between the points $z \in D$ in $\mathbb{C}$ or $\mathbb{C}_z$ and the points $w = f(z)$ that belong to part of $\mathbb{C}'$ or $\mathbb{C}_w$ is called a mapping from the z-plane to the w-plane (also a transformation) of the domain D by f into $\mathbb{C}'$ or $\mathbb{C}_w$. However, if R is the range of f i.e. the set of points $w = f(z)$ in $\mathbb{C}'$ or $\mathbb{C}_w$ with $z \in D$, the correspondence is called the mapping of D by f onto R.

The point $w_0 = f(z_0)$ (corresponding to $z \in D$) is called the image point in the w-plane of the point z_0 in the z-plane under the mapping f. Otherwise the mapping function f is specified, the point w_0 is called the image of z_0 or z_0 is called the pre-image of w_0.

Now ~~onto~~ into and onto

In a mapping by f of D into a set S, not every point of S is the image of a point in D, whereas in the case of mapping of D onto a set R, every point of R is the image of at least one point in D. It is illustrated in figure below.



There will be a class of mappings f for which the relation

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2$$

These are called one-one mappings because they have the property that distinct ~~z~~ points z_1 and z_2 have distinct images w_1 and w_2. When a mapping by f of D is both onto and one-one i.e one z corresponds to one w and conversely then inverse function exists that maps any point w back to a point z in a unique way. The function f is usually denoted by f^{-1} (not to be confused with $1/f$) with that if $w = f(z)$ then $z = f^{-1}(w)$. If domain D is mapped onto the range R of f one-one, then the inverse function f^{-1} maps R one-one onto D.

Geometrical Representation:

To draw a curve of complex variable (x, y) on Z-plane we take two axes i.e. one real axis and the other imaginary axis. A number points (x, y) are ploted on Z plane by taking different values of Z (different values of x and y). The curve C is drawn by joining the ploted points. The diagram is called Argand diagram in Z-plane.

But a complex function $w = f(Z)$ i.e. $u + iv = f(x+iy)$ involves four variables x, y and u, v. A figure of only three dimensions (x, y, z) is possible in a plane. A figure of 4 dimensional region for x, y and u, v is not possible. So we choose two complex planes Z-plane and w-plane. In the w-plane we plot the corresponding points $w = u + iv$, by joining these points we have a corresponding curve C in w-plane.

Transformation: or mapping

For every point (x, y) in the Z-Plane, the relation $w = f(Z)$ defines a corresponding point (u, v) in the w-plane, we call this "transformation or mapping of Z-plane into w-plane". If a Point Z_0 maps into the w_0 (from Z-Plane to w-plane). The w_0 is known as the image of Z_0.

If the Point $P(x, y)$ moves along a curve C in Z plane the Point $P'(u, v)$ will move along along a corresponding curve C' in w-plane, then we say that a curve C in the Z plane is mapped into the corresponding curve C' in the w-plane by the relation $w = f(Z)$.

EX (1) Transform the rectangular region ~~ABCD~~ ABCD in z-plane bounded by $x=1$, $x=3$, $y=0$ and $y=3$. under transformation (mapping) $w = z+(2+i)$

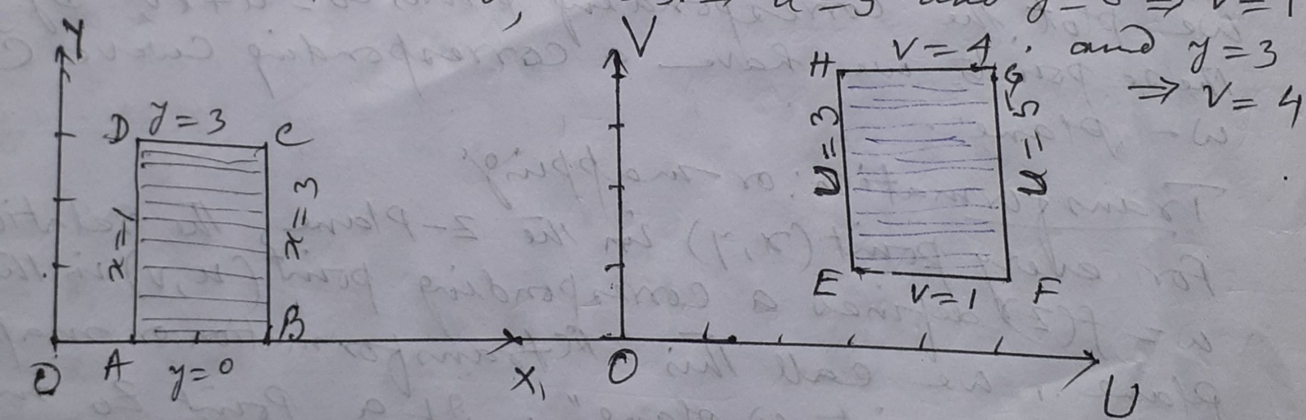
Sol^n: Here $w = z+(2+i)$

i.e. $u+iv = x+iy+(2+i) = (x+2)+i(y+1)$, we have equating real and imaginary parts,

$u = x+2$. and $v = y+1$.

Z plane	w plane	Z plane	w-Plane
x	$u=x+2$	y	$v=y+1$
1	$=1+2=3$	0	$=0+1=1$
3	$=3+2=5$	3	$=3+1=4$

$x=1 \Rightarrow u=3$, $x=3 \Rightarrow u=5$ and $y=0 \Rightarrow v=1$ and $y=3 \Rightarrow v=4$



Here the lines $x=1$, $x=3$, $y=0$ and $y=3$ in z-plane are transformed onto the line $u=3$, $u=5$, $v=1$ and $v=4$ in the w-plane. The Region ABCD in z plane transformed into the region EFGH. in w-plane.

EX. Transform the curve $x^2-y^2=4$ under mapping $w=z^2$

Hints:- $w=z^2=(x+iy)^2$, $u+iv=x^2-y^2+2ixy$.

This gives $u=x^2-y^2$, and $v=2xy$.

Prepare table of (x,y) and (u,v) s.t.

x	2	2.5	3	3.5	etc
$y=\pm\sqrt{x^2-4}$	0				-
$u=x^2-y^2$	4				-
$v=2xy$	0				-

Plot (x,y) in z-plane and then find the points corresponding (u,v) in the w-plane. You will see image of curve $x^2-y^2=4$ is a st. line $u=4$ parallel to the v-axis in w plane.

Conformal transformation: Let two curves C and C_1 in the z-plane intersect at the point P and the corresponding curve C', C_1' in the w-plane intersect at P'. If the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curves of w-plane at P' in magnitude and sense then the transformation or mapping is called conformal.

Condition (i) $f(z)$ is analytic (ii) $f'(z)\neq 0$.

If the sense and magnitude of the angle is preserved the transformation is said to be conformal.

If only the magnitude of the angle is preserved then transformation is Isogonal.

Note: ① The point at which $f'(z)=0$ is called a critical point of the transformation also the points where $\frac{dw}{dz} \neq 0$ are called ordinary points.

② It is stated that if the tangent at P to the curve is rotated through an angle ϕ which is equal to amp $\{f'(z)\}$ under the given transformation.
and Angle of rotation is $\tan^{-1}\left(\frac{v}{u}\right)$ [ie $\tan\phi = \frac{dy}{dx}$, if ϕ is angle of intersection of the curves.

③ In formal transformation, the element of ~~are~~ arc passing through P is magnified by the factor $|f'(z)|$ the area element is also magnified by the factor $|f'(z)|$ or $J = \frac{\partial(u,v)}{\partial(x,y)}$ in a Conformal transformation.

and hence $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

$= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2 = |f'(x+iy)|^2$ by C-R theory as $f(z)$ is analytic

and $|f'(z)|$ is called the coefficient of magnification.

④ Conjugate function remain conjugate function after Conformal transformation. A function which is solution of Laplace's equation, its transformed function again remains the solution of Laplace's equation after Conformal transformation.

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Ex ③ If $u = 2x^2 + y^2$ and $v = \frac{y^2}{x}$ if the curve u, v are constant then show that they cut orthogonally at all intersection but that the transformation $w = u + iv$ is not conformal.

Solⁿ: Let $u = 2x^2 + y^2 = k_1$ (say constant) ①

then $4x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$ [differentiated] ②

for the curve $v = \frac{y^2}{x} = k_2$ (say) ③ $\Rightarrow y^2 = k_2 x$.

$\Rightarrow y^2 = k_2 x \Rightarrow 2y\frac{dy}{dx} = k_2 \Rightarrow$ ~~$\frac{dy}{dx} = \frac{k_2}{2y}$~~ ④

$\frac{dy}{dx} = \frac{k_2}{2y} = \frac{y^2}{x} \times \frac{1}{2y} = \frac{y}{2x}$

from ② & ④ $\tan\phi_1 = \frac{dy}{dx} = -\frac{2x}{y}$ & $\tan\phi_2 = \frac{dy}{dx} = \frac{y}{2x}$

hence $m_1 \times m_2 = -1$ Hence two curves cut orthogonally. However we see that

$\frac{\partial u}{\partial x} = 4x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}$, $\frac{\partial v}{\partial y} = \frac{2y}{x}$.

Cauchy-Riemann equation are not satisfied by u & v. Hence the function $u + iv$ is not analytic so the transformation is not conformal.

Ex ④ For the conformal transformation $w = z^2$ show that

a) The co-efficient of magnification at $z = 2+i$ is $2\sqrt{5}$

b) The angle of rotation at $z = 2+i$ is $\tan^{-1}(0.5)$

Solⁿ: Let $w = f(z) = z^2 \Rightarrow f'(z) = 2z \Rightarrow f'(2+i) = 2(2+i) = 4+2i$

a) Co-efficient of magnification at $z = 2+i$ is $|f'(2+i)| = |4+2i| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

b) Angle of rotation at $z = 2+i$ is $\text{amp}\, f'(2+i) = \text{amp}(4+2i) = \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}(0.5)$

Ex do yourself: for conformal transformation $w = z^2$ find i) the co-efficient of magnification $z = 1+i$ is $2\sqrt{2}$

ii) the angle of rotation at $z = 1+i$ is $\pi/4$.