Class notes by Amalendu Sekhar Pattanayak.
For Sem-6(H) Core-13 Paper C13T
Unit-4. Complex Analysis

NB: Complex definite integrals are called (complex) line integrals they are written $\int_C f(z)\,dz$.. The curve C in the complex plane is called the path of integration, the curve is for simplicity. we may represent by a parametric representation.

$$z(t) = x(t) + iy(t) \quad \text{—(1)} \quad (a \le t \le b).$$

For instance $z(t) = t + 3it$ $(0 \le t \le 2)$ gives a portion of the line $y = 3x$, here $x = t$ (Let) the for $x+iy$, $y = 3x = 3t$.

Now for $z(t) = 4\cos t + 4i\sin t$ $(-\pi \le t \le \pi)$ and represents the circle $|z| = 4$.

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t) \quad \text{—(2)}$$

Let C be a piecewise smooth path, represented by $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be continuous on C then

$$\int_C f(z)\,dz = \int_a^b f\{z(t)\}\dot{z}(t)\,dt \quad \dots (3).$$

we have $z = x + iy \Rightarrow \dot{z} = \dot{x} + i\dot{y}$

if we write $u = u\{x(t), y(t)\}$ and $v = v\{x(t), y(t)\}$

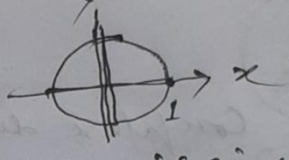
so $dx = \dot{x}\,dt$, $dy = \dot{y}\,dt$ then

$$\int_a^b f\{z(t)\}\dot{z}(t)\,dt = \int_a^b (u+iv)(\dot{x}+i\dot{y})\,dt$$

$$= \int_C \{(u\,dx - v\,dy) + i(u\,dy + v\,dx)\}$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

~~Imp~~ basic result: ~~Int~~ Integral of $\frac{1}{z}$ around the unit circle



$$\oint_C \frac{dz}{z} = 2\pi i$$

$z(t) = \cos t + i \sin t$ (unit circle. $0 \le t \le 2\pi$)

$= e^{it}$ ... ①

$\dot{z}(t) = i e^{it}$ integrand corresponds to an increase of t from 0 to 2π (counter clockwise)

here ~~$f(z)$ $f\{z(t)\}$~~ $f\{z(t)\} = \frac{1}{z(t)} = e^{-it}$ by ①

Hence $\int \frac{dz}{z} = \left[\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)\,dt\right.$

$$= \int_0^{2\pi} e^{-it} \cdot i e^{it} = i\int_0^{2\pi} dt = 2\pi i$$

ML – inequality

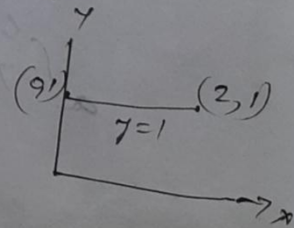
For absolute value of Complex line integral

$\left|\int_C f(z)dz\right| \le ML$ where L is the length of the curve and ~~At~~ M a +ve constant such that $|f(z)| \le M$, on C.

* EX ① Let $f(z) = \frac{1}{z^2}$ and Γ be the st. line joining the points i and $i+2$ show that $\left|\int_\Gamma f(z)\,dz\right| \le 2$

Sol^n: on Γ $|f(z)| = \left|\frac{1}{z^2}\right| = \frac{1}{|z|^2} = \frac{1}{x^2+y^2}$

$= \frac{1}{x^2+1} \le 1 = M$ (let) ... ①



[Now use $\left|\int_\Gamma f(z)dz\right| \le ML$] Here length $L = 2$

so ~~ML~~ ≤ 2 or, $\left|\int_\Gamma f(z)dz\right| \le 2$

Cauchy's Fundamental or integral Theorem:

Th: If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, Then $\int_C f(z)\,dz = 0$.

Proof: Let The region enclosed by the curve C be R and let $f(z) = u+iv$, $z = x+iy$, $dz = dx+idy$

Then $$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx+udy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

by Green's Theorem.

by C-R equation replacing $-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ Then we get

$$\int_C f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)dxdy$$

$$= 0 + i0 = 0$$

$$\Rightarrow \int_C f(z)\,dz = 0$$

Note ① If there is no pole inside and on The Contour Then The value of integral of the function is Zero.

Note ② Poles of the integrand is when the denominator of The integrand becomes Zero.

Cauchy's extension Theorem to multiple connected regions: If C is a simple closed curve and $C_1, C_2, \ldots C_n$ are non intersecting simple closed curves which are in C Then

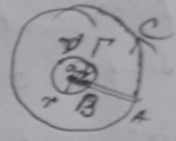
$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_n} f(z)\,dz.$$

N.B. Cauchy integral Theorem does not apply to domain D with piecewise smooth Contours, such as semicircle, rectangles, but later Goursat showed This Theorem to be applicable to domain with piece wise smooth boundaries This extension is extremely important because of Goursat Contribution the Theorem is called Cauch- Goursat Theorem.

V.V.Imp. Cauchy's integral formula:

Th: Let $f(z)$ be analytic within and on a simple closed curve C and let a be any point inside C

Then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$.



Proof. We enclose the point $z=a$ by a small circle Γ of radius r lie entirely within curve C, joining the contour C and the circle Γ by a narrow cut AB. We construct the function $\frac{f(z)}{z-a}$, which is analytic in the closed annulus (circular ring) bounded by C and Γ

Hence $\oint_C \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)}{z-a}\,dz = \oint_\Gamma \left[\frac{f(z)-f(a)+f(a)}{z-a}\right] dz$

$$= \oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + \oint_\Gamma \frac{f(a)}{z-a}\,dz \quad \text{---} (1)$$

Since Γ is a circle of radius r with centre at point a

Then $|z-a| = r$ .... (2) $\Rightarrow$ $z-a = re^{i\theta}$ .... $0 \le \theta \le 2\pi$ (3)

$\therefore dz = rie^{i\theta}d\theta$ ... (4)

Now $\oint_\Gamma \frac{f(a)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a)}{re^{i\theta}}\, rie^{i\theta}d\theta = \int_0^{2\pi} i f(a)\,d\theta$

$$= if(a)\int_0^{2\pi} d\theta = 2\pi i f(a) \quad \text{-----} (5)$$

again $\left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz\right| \le \frac{1}{2\pi}\oint_\Gamma \left|\frac{f(z)-f(a)}{z-a}\right| |dz|$ as $|i| = 1$

$$= \frac{1}{2\pi}\oint_\Gamma \frac{|f(z)-f(a)|\,|dz|}{|z-a|} \quad (6) \text{ by } (2) \,\&\, (4)$$

Since the integrand $\left|\frac{f(z)-f(a)}{z-a}\right|$ is analytic except at $z=a$ for very small $\epsilon > 0$ $\exists\ \delta > 0$ s.t.

$|f(z)-f(a)| < \epsilon$ for all z in disk $|z-a| < \delta$ and (2) $|z-a| = r$

and $|dz| = |rie^{i\theta}d\theta| = r$ from Relⁿ (6) we get

$$\left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)-f(a)}{z-a}\right| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{\epsilon\,|dz|}{|z-a|} = \frac{\epsilon}{2\pi}\int_0^{2\pi} \frac{r\,d\theta}{r} = \frac{\epsilon}{2\pi}\cdot 2\pi = \epsilon \to 0 \quad (7)$$

ϵ is arbitrary

N.B. $|e^{i\theta}| = 1$ Therefore from (1) we have using (7) and (5)

$$\oint_C \frac{f(z)\,dz}{z-a} = \oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + \oint_\Gamma \frac{f(a)}{z-a}\,dz$$

$$= 0 + 2\pi i f(a) \Rightarrow f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

This proves the theorem.

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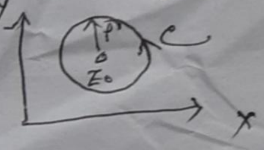
NB. Independence of Path: If $f(Z)$ is analytic in a simply connected domain D, then the integral of $f(Z)$ is independent of Path in D. ~~A~~

An integral of $f(Z)$ independent of Path in a domain D if for every Z_1, Z_2 in D its value depends only on the initial Point Z_1 and the terminal point Z_2, but not of the choice of the path C in D [so that every path in D from Z_1 to Z_2 gives the same value of the integral of $f(Z)$.]

Example: Integral of $\frac{1}{Z^m}$ with integer Power m.

Let $f(Z) = (Z - Z_0)^m$ where m is the integer and Z_0 a constant. integrate counterclockwise around the circle C of radius ρ with centre at Z_0



Solⁿ: we may represen C in the form

$Z(t) = Z_0 + \rho(\cos t + i\sin t) = Z_0 + \rho e^{it}$ so we have

$(Z - Z_0)^m = \rho^m e^{imt}$, $dZ = i\rho e^{it}dt$ and $(0 \le t \le 2\pi)$

$$\oint_C (Z-Z_0)^m dZ = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} dt = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt$$ by Euler formula

$$= i\rho^{m+1}\left[\int_0^{2\pi}\cos(m+1)t\,dt + i\int_0^{2\pi}\sin(m+1)t\,dt\right.$$ [If $m = -1$, $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$ ~~thus~~ value is $2\pi i$, For $m \neq -1$ each of two integral is zero.

hence $$\oint_C (Z - Z_0)^m dZ = \begin{cases} 2\pi i & (m = -1 \\ 0 & (m \neq -1, \text{ and integer}) \end{cases}$$

NB

a complex line integral depends not only on the endpoints of the Path but in general also on the Path itself.