For 6th Sem (H)
Complex Analysis
Paper — C13T unit — 4

Exponential, Logarithm function etc.

A function $f(z)$ is said to be analytic in a Domain D if $f(z)$ is defined and differentiable at all points of D. The function is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighbourhood of z_0.

The term analytic in D is holomorphic in D.

The non negative powers $1, z, z^2 \ldots$ are analytic in the entire complex plane and so the Polynomial function of the form ~~$f(z) = c_0 + c$~~

$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$ where $c_0, c_1, \ldots$ are complex constant are analytic.

The quotient of two polynomials ~~$g(z)$ and $f(z)$~~ $g(z)$ and $h(z)$ ie $f(z) = \dfrac{g(z)}{h(z)}$ (rational function). This f is analytic except at the points where $h(z) = 0$ possibility of common factors of $g(z)$ & $h(z)$ maybe there.

① The most important analytic function, the complex exponential function e^z also written as $\exp z$.

Prop-① Now $e^z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i \sin y)$

Since $e^z = u + iv$ we set $u = e^x \cos y$ & $v = e^x \sin y$ (check Cauchy-Riemann eqn.) ie

$u_x = e^x(-\sin y) + e^x \cos y$ and ~~$u_y = ?$~~ $u_y = ?$ find

$v_x = e^x \cos y + e^x \sin y$ and $v_y = ?$ find

it will be seen that it will be satisfied the 1st order partial derivatives are continuous showing that e^z is analytic for all z, and so e^z is an entire function.

Prop: 2 Show $\dfrac{d}{dz}(e^z) = e^z \Rightarrow \dfrac{d}{dz}(e^z) = \dfrac{du}{dx} + i\dfrac{dv}{dx} = (e^x \cos y)_x + i(e^x \sin y)_x$

$= e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = e^x . e^{iy} = e^{x+iy} = e^z$.

$\therefore (e^z)' = e^z$ Verified

Again we know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

replace x by z, $\Rightarrow e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}$. which is a polynomial hence it is analytic.

so e^z is entire function at analytic for all z.

Prop: 3.

$e^{z_1+z_2} = e^{z_1} e^{z_2}$ holds for $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

Now
$$e^{z_1} e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2)$$
$$= e^{x_1+x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2)$$
$$= e^{x_1+x_2}[\cos(y_1+y_2) + i \sin(y_1+y_2)]$$
$$= e^{z_1+z_2}$$

∴ If we put $z_1 = x$ & $z_2 = iy$ then special case
$$e^z = e^{x+iy} = e^x e^{iy} = \cancel{e^{z_1} e^{z_2}}$$

∴ Polar form of complex no. $z = r(\cos\theta + i \sin\theta)$
i.e. $z = re^{i\theta}$.

∴ we know $e^{2\pi i} = 1 = \cos 2\pi + i \sin 2\pi$ & $e^{\frac{\pi i}{2}} = i$ etc.

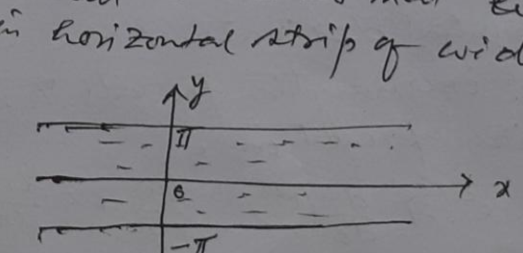
∴ $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$

so $|e^z| = |e^x(\cos y + i \sin y)| = e^x \cdot 1$, and hence
$$\arg e^z = \tan^{-1}(\tan y) \pm 2\pi i n \quad (n = 0, 1, 2 \dots)$$
$$= y + 2n\pi i$$

here it shows that $|e^z| = e^x \neq 0$, never vanishes.

periodicity of e^z with $2\pi i$: $e^{z+2\pi i} = e^z$ for all z.

Hence all the values that $w = e^z$ can assume in horizontal strip of width 2π. i.e.



Fundamental region of the exponential function e^z in z-plane.

EX ① Take the Product of

$e^{2+i} = e^2(\cos 1 + i\sin 1)$ and $e^{4-i} = e^4(\cos 1 - i\sin 1)$ #

$\Rightarrow e^{2+i} \cdot e^{4-i} = e^2 \cdot e^4 (\cos^2 1 + \sin^2 1) = e^6.$

verified.

EX ② Solve the equation $e^z = 3+4i$, find z.

Sol^n. $|e^z| = |3+4i| = \sqrt{3^2+4^2} = 5$

again $|e^z| = |e^x(\cos y + i\sin y)| = e^x \cdot \sqrt{\cos^2 y + \sin^2 y}$

$\therefore e^z \quad = e^x$

Hence $e^x = 5 \Rightarrow x = \log 5 = 1.609 \cdots$ — ①

Now $e^z = \cancel{3+4i} = e^x(\cos y + i\sin y) = 3+4i$

$\Rightarrow \quad e^x \cos y = 3; \; e^x \sin y = 4$.

$\therefore \cos y = \frac{3}{5} = 0.6$ and $\sin y = \frac{4}{5}$, $y = \tan^{-1}\left(\frac{4}{3}\right)$

$y = 0.927$ — ② Hence $z = x + iy = 1.609 + 0.927i \pm 2\pi i n$

$n = 0, 1, 2 \cdots$

$\therefore z$ has infinitely many solution for the Periodicity of e^z.

EX. ① Do yourself find z ~~when $e^z = 4-3i$~~

when ① $e^z = 4-3i$. ② $e^z = -2$

~~② $e^z = 2$~~.

③ write in polar form $\sqrt{i}$

Sol^n. $\sqrt{\frac{2i}{2}} = \sqrt{\frac{(1+i)^2}{2}} = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$x = \frac{1}{\sqrt{2}}, \; y = \frac{1}{\sqrt{2}} \; \therefore r = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$. Hence the required

Polar form $re^{i\theta} = 1 \cdot e^{\frac{\pi i}{4}} = e^{\frac{\pi i}{4} + 2\pi i}$

④ Do yourself: write the Polar form of -9

Ans $\underline{9e^{\pi i}}$

Important: It may be noted that e^{∞} has no meaning in the complex plane.

Now Complex Trigonometric function.

by Euler formulas

$e^{ix} = \cos x + i\sin x$, $e^{-ix} = \cos x - i\sin x$

by addition; $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ for Complex values

$z = x+iy$ these are $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ (1)

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ etc.

Since e^z is entire, $\cos z$, $\sin z$ are entire functions but $\tan z$ and $\sec z$ are not entire, they are analytic except at the points where $\cos z = 0$, so are the $\cot z$ and $\operatorname{cosec} z$.

For derivatives $(e^z)' = e^z$ (as early shown)

So $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$, $(\tan z)' = \sec^2 z$ etc

Eqn (1) also shows that Euler's formula is valid for complex. $\therefore e^{iz} = \cos z + i\sin z$

Real and imaginary parts of $\cos z$ and $\sin z$ to be needed in computing values.

Formula: $\cos z = \cos x\cosh y - i\sin x\sinh y$.

$\sin z = \sin x\cosh y + i\cos x\sinh y$.

$|\cos z|^2 = \cos^2 x + \sinh^2 y$; $|\sin z| = \sin^2 x + \sinh^2 y$ etc.

To prove this use, $\cos z = \frac{1}{2}\left\{e^{i(x+iy)} + e^{-i(x+iy)}\right\}$

$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$

$= \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{1}{2}i(e^{y} - e^{-y})\sin x$

$= \cosh y\cos x - i\sinh y\sin x$

[as $\frac{1}{2}(e^{y} + e^{-y}) = \cosh y$, $\sinh y = \frac{1}{2}(e^{y} - e^{-y})$

etc.

Ex

Solve $\cos z = 5$ (which has no real solution but complex solution.

Soln: $\frac{1}{2}(e^{iz} + e^{-iz}) = 5 \Rightarrow e^{2iz} - 10e^{iz} + 1 = 0$

multiplying by e^{iz} both sides

Next Page

$e^{2iz} - 10e^{iz} + 1 = 0$

This is a quadratic eqn in e^{iz}.

so $e^{iz} = 5 \pm \sqrt{25-1} = 5 \pm \sqrt{24} = e^{-y+ix}$

or, $e^{iz} = e^{-y}(e^{+ix}) = e^{-y}(\cos x \pm i \sin x) = 5 \pm 2\sqrt{6} = 9.899 \pm i0$ or 0.101

$e^{-y}(\cos x \pm i \sin x) = 9.899$ or 0.101 — ①

$\Rightarrow e^{-y}\cos x = 9.899$ or 0.101 — ①

$e^{-y}\sin x = 0$ — ②

$①^2 + ②^2 \Rightarrow e^{-2y} = (9.899)^2$

$\therefore e^{-y} = 9.899$ or 0.101 $\Rightarrow -y = \log_e(9.899) = 2.292$

or $-y = -2.292$

hence $y = \pm 2.292$ $(n = 0, 1, 2 \ldots$

and $\cancel{x =}$ $\sin x = 0 \Rightarrow x = \pm 2n\pi$

hence $z = x + iy = \pm 2n\pi \pm i\,2.292$

Solve for 1) $\cos z = 0$

Soln. $\frac{1}{2}(e^{iz} + e^{-iz}) = 0 \Rightarrow e^{2iz} + 1 = 0$

$\therefore e^{2iz} = -1 = i^2 \Rightarrow e^{iz} = \pm i$

$\Rightarrow e^{i(x+iy)} = e^{-y} \cdot e^{ix} = e^{-y}(\cos x + i\sin x) = 0 \pm i$

$\Rightarrow e^{-y}\cos x = 0$, $e^{-y}\sin x = \pm 1$

squaring and adding $e^{-2y}(1) = 1$

$\Rightarrow e^{y} = e^{0} \Rightarrow y = 0$

hence $e^{-y}\cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm(2n+1)\pi/2$

$n = 0, 1, 2$.

hence $z = x + iy = \pm\frac{1}{2}(2n+1)\pi$ $(n = 0, 1, 2 \ldots$

Solve 2) $\sin z = 0$ do yourself

Logarithm function (Complex) Referrence Book Allan Jeffrey

The natural logarithm of $z = x+iy$ is denoted by $\ln z$ if we take $\omega = \ln z$ where $z \neq 0$. then $e^{\omega} = z$ (Note that $z=0$ impossible since $e^{\omega} \neq 0$.)

If we set $\omega = u+iv$ and $z = re^{i\theta}$ then

$$e^{\omega} = e^{u+iv} = e^{u} \cdot e^{iv} = r \cdot e^{i\theta} \quad \text{—(1)}$$

Now $|z| = |e^{\omega}| = |e^{u+iv}| = |e^{u}(\cos v + i\sin v)| = \sqrt{(e^{u}\cos v)^2 + (e^{u}\sin v)^2}$
$= e^{u}$

hence from (1) $e^{u} = r = |z|$ showing that $u = \ln_e |z|$ and $v = \theta$. $(\sqrt{x^2+y^2})$,

Now $w = u+iv = \ln z$ is given by

$$= \ln(e^{\omega}) = \ln(re^{i\theta})$$
$$= \ln r + i\theta \quad \text{(2)} \qquad r = |z| > 0, \quad \theta = \arg z.$$

The complex natural logarithm $\ln z$ $(\log_e z)$ $(z \neq 0)$ is infinitely many valued.

~~General value~~ $v = \theta = \arg(z) = \cancel{Arg(z)+2\pi i}$
$= Arg(z) + 2n\pi$ $(n = 0, \pm 1, \pm 2 \dots$

where $\cancel{<Arg(z)}$ $-\pi < Arg(z) \leq \pi$.

Now combining $u = \ln_e |z|$ and $v = \theta = Arg(z) + 2n\pi$
we get $\ln z = \underbrace{\ln_e |z|}_{= \ln r} + i\{Arg(z) + 2\pi n\}$ $\quad$ (3) $\quad n = 0, \pm 1, \pm 2 \dots$

$\ln z$ is infinitely many valued for any z, with all real part of $\ln z$ the same but imaginary parts differ by integral multiple of 2π.

The Principal part of $\ln z$ is denoted by $\mathrm{Ln}\, z$ $n=0$

$\therefore$ $\mathrm{Ln}\, z = \ln |z| + i\,Arg(z)$ (for $z \neq 0$) ~~general value~~

Hence $\omega = \ln_e |z| + i\,Arg(z) + 2n\pi i$ $(n = 0, \pm 1, \pm 2$
General value

Referrence: Allan Jeffrey.

If $z = x + iy$

then $w = Ln z \Rightarrow e^{w} = z$.

then $Ln z = ln_e[(x^2+y^2)^{1/2}] + i \arg(z) \pm 2n\pi i$

$= \frac{1}{2} ln(x^2+y^2) + i \tan^{-1}(y/x) \pm 2n\pi i$

for derivative

$$\frac{d}{dz}(ln z) = \frac{1}{2}\frac{\partial}{\partial x} ln(x^2+y^2) + i\frac{\partial}{\partial x}(\tan^{-1} y/x)$$

$$= \frac{1}{2}\frac{2x}{(x^2+y^2)} + i\frac{-y/x^2}{1+(\frac{y}{x})^2} =$$

$$= \frac{x}{(x^2+y^2)} + i\frac{(-y)}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{(x-iy)}{(x-iy)(x+iy)} = \frac{1}{x+iy} = \frac{1}{z}.$$

Example ① $ln(-1)$ & $Ln(-1)$

As $e^{i\pi} = -1$

$\therefore ln(-1) = ln|e^{\pi i}| + i\{Arg(-1) + 2n\pi\}$

for principal value $n=0$.

$= ln|-1| + i(\pi + 2n\pi)$, for $n = 0, \pm1, \pm2, \ldots$

$= (1+2\times0)\pi i$ for principal value $n=0$

$= \pi i$

and $Ln(-1)$

Ex Find z^i, when $z = \frac{3}{\sqrt{2}}(1+i) = 3e^{i\frac{\pi}{4}}$. then

we see that $|z| = 3$ and $Arg z = \frac{\pi}{4}$.

$z^i = e^{i(Ln z)} = e^{[i(ln_e 3 + \frac{1}{4}\pi i + 2n\pi i)}$ for $n = 0, \pm1, \pm2, \ldots$

$= e^{-\frac{1}{4}\pi(1+8n) + i\, ln_e 3} = \exp\left[-\frac{1}{4}(1+8n)\pi\right](\cos ln_e 3 + i \sin(ln_e 3))$

This has infinitely many values

Principal value is for $n=0$ and hence

$z^i = e^{-\pi/3}(\cos ln_e 3 + i(\sin ln_e 3))$