

Class notes by Amalansu Sekhar Patanayak
for Math (H) SEM-6 students
Complex Analysis (Harmonic function)

Any function which satisfies Laplace's equation is known as harmonic function.

i.e. the solutions of Laplace's eqnⁿ are the harmonic function.

The most independent PDE in the expression

$\nabla^2 u$ is called Laplacian of u
Laplacian equation in space $\nabla^2 u = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{Two dimensional}$$

Laplacian equation for steady (time-independent) Problem ($\frac{\partial u}{\partial t} = 0$)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Th. If $f(z) = u + iv$ is an analytic function then u and v are both harmonic function. as u, v are harmonic so u, v are the solutions of Laplace's eqnⁿ. so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

such ~~the~~ functions u, v are called Conjugate harmonic functions

Example: Find the harmonic Conjugate function of the function $u(x, y) = 2x(1+iy)$

Solⁿ: $f(z) = u + iv$ as u, v harmonic conjugate function so $f(z)$ is analytic and satisfy C-R equation (Cauchy-Riemann equation).

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we have $u(x, y) = 2x(1-y)$ Let v be the harmonic of u . By total differentiation

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{by C-R Condition}$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -(-2x)dx + (2-2y)dy$$

$$dv = 2x dx + 2(1-y) dy$$

integrating

$$v = x^2 + 2y - y^2 + C \quad \text{and hence}$$

the harmonic conjugate of u is $x^2 + 2y - y^2 + C$.

EX. Show that the function $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic. Find its harmonic conjugate.

Solⁿ. $u = \frac{1}{2} \log(x^2 + y^2)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2} \quad ; \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\text{and } \frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{u satisfies the Laplace's eqn.}$$

So u is a harmonic function

Let v be the conjugate function of u , by total differential

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{by (C-R) Condition}$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\text{or } dv = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \quad \text{integrating}$$

$$= \frac{xdy - ydx}{x^2 + y^2} = d \tan^{-1}(y/x) \quad \text{integrating}$$

$$v = \tan^{-1} \frac{y}{x} + C \quad [C \text{ is real const.}]$$

This is the required conjugate.

EX. Given $f(z) = u(x, y) + iv(x, y)$ be an analytic function and $u = 3x - 2xy$, then find v and express $f(z)$ in terms of z .

Sol. we have $u = 3x - 2xy$

$$\therefore \frac{\partial u}{\partial x} = 3 - 2y, \quad \frac{\partial u}{\partial y} = -2x$$

By total differential

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

by C-R equation

$$\text{as } u = 3x - 2xy$$

$$dv = -2x dx + (3 - 2y) dy$$

integrating

$$v = \int 2x dx + \int (3 - 2y) dy = x^2 + 3y - y^2 + c$$

$$\therefore f(z) = u(x, y) + iv(x, y) = (3x - 2xy) + i(x^2 + 3y - y^2 + c)$$

$$= (ix^2 - iy^2 - 2xy) + 3x + 3yi + ic$$

$$= i(x + iy)^2 + 3(x + iy) + ic$$

$$= iz^2 + 3z + ic \quad \text{which is the required expression.}$$

or put $y=0$ and $x=z$ in $\text{Re} f(z)$ ①

$$\text{hence } f(z) = 3z + iz^2 + ic$$

EX. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v s.t. $f(z) = u + iv$ is analytic. Also express in terms of z .

Sol. we have $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2y - 2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \quad \text{Since Laplace eqn is satisfied}$$

u is harmonic

Now apply total differential

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

By C-R eqn.

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{--- ① Putting the values}$$

$$= -(-2y - 2x + 3) dx + (2x - 2y - 2) dy$$

integrating

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$$v = 2xy + x^2 - 3x - y^2 - 2y + c$$

now $f(z) = u + iv$

$$= (x^2 - y^2 - 2xy - 2x + 3y) + i(2xy + x^2 - 3x - y^2 - 2y)$$

$$= (x+iy)^2 + i(x+iy)^2 - (2+3i)(x+iy) + ic$$

$$f(z) = z^2 + iz^2 - (2+3i)z + ic$$

or Put $y=0$ & $x=z$ in ① we get the result

EX. Find ~~A and B~~ ^{A and B} so that

$f(x,y) = x^2 + Axy + By^2$ is harmonic

Solⁿ. we have $\frac{\partial f}{\partial x} = 2x + Ay$ and $\frac{\partial^2 f}{\partial x^2} = 2$ --- ①

and $\frac{\partial f}{\partial y} = Ax + 2By$ and $\frac{\partial^2 f}{\partial y^2} = 2B$. --- ②

Adding ① and ② we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 + 2B \quad \text{for harmonic } f^h.$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow 2 + 2B = 0 \Rightarrow B = -1$$

so $f(x,y) = x^2 + Axy - y^2 = u(x,y)$ (let it be $u(x,y)$)

now $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ by C-R equation

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -(Ax - 2y) dx + (2x + Ay) dy$$

$$\Rightarrow v = -\frac{Ax^2}{2} + 2d(xy) + \frac{Ay^2}{2} = -\frac{Ax^2}{2} + 2xy + \frac{Ay^2}{2}$$

now $\frac{\partial v}{\partial x} = -Ax + 2y$, $\frac{\partial^2 v}{\partial x^2} = -A$.

$\frac{\partial v}{\partial y} = 2x + Ay$, $\frac{\partial^2 v}{\partial y^2} = A$ so that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

It is seen that for any Real value of A will satisfy.