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for SEM-6(H) Complex Analysis
Unit-4. Paper C13T Core-13.

P-I

Complex integration

In case of real variable the Path of Integration of $\int_a^b f(x)dx$ is always along the x-axis from a to b. But in case of complex function $f(z)$ the Path of definite integral $\int_a^b f(z)\,dz$ is ~~any~~ along any curve from $z=a$ to $z=b$.

$z = x+iy \Rightarrow dz = dx + i\,dy$ — (1) $dz = dx$ if $y=0$ — (2)

$dz = i\,dy$ if $x=0$ (3) In (1), (2), (3) the direction of dz are different. Its value depends upon the path of integration. But the value of integration from a to b remains the same along any regular curve from a to b. The contour integral is denoted ~~If~~ by $\oint_C f(z)dz$.

If $f(z) = u(x,y) + iv(x,y)$ then

$$\oint_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Continuous arc:

If a point z on an arc such that

$z = \phi(t) + i\psi(t)$ --- (1)

then we may write $x = \phi(t)$ --- (2),
$y = \psi(t)$ --- (3).

If $\phi(t)$ and $\psi(t)$ are real continuous function of the real variable t, defined in range $a \le t \le b$, then the arc is called continuous arc.

Multiple point:

If the equation (1) or (2) and (3) are satisfied by more than one value of t in the given range then the point z is a multiple of arc.

* **Jordan arc:**

A continuous arc without multiple point is called Jordan arc.

In addition if $\phi'(t)$, $\psi'(t)$ are also continuous in the given range $a \le t \le b$ then the arc is called a regular arc of a Jordan arc.

* Contour :- If the initial point and final point coincide of the curve so that C is closed. in this Integral is called contour integral denoted by $\oint_C f(z)\,dz$

* **Contour:**

Contour means a continuous chain of Jordon curve consists of finite number of regular arcs.

Alternative: A region R is said to be simply connected if every closed curve lying within it encloses only points of the region.

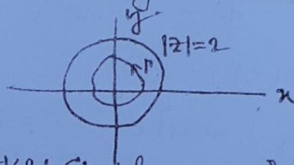
Closed contour:

A simple closed curve is called a closed contour if it consists of a finite number of regular arcs.

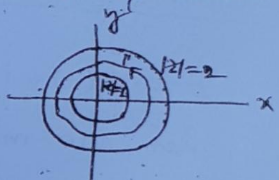
Simply and Multiply Connected Regions:

A domain or a region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R.

A region R which is not simply connected is called multiply connected.

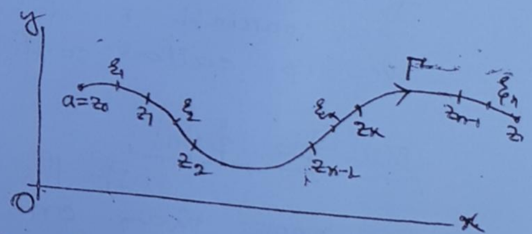


$|z|<2$; Simply connected



$1<|z|<2$; Multiconnec

Complex Line Integral:



Let $f(z)$ be continuous at all points of a Γ having finite length. Divide Γ into n parts by means of points $z_1, z_2, \dots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$.

We choose a point ξ_r on each arc joining z_{r-1} to z_r and form the sum $S_n = \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$.

We assume the maximum value of $(z_r - z_{r-1}) \to 0$ as $n \to \infty$ so that $S_n \to$ finite limit which does not depend upon the mode of subdivision and we denote this limit by $\int_a^b f(z)dz$ or $\int_\Gamma f(z)dz$.

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Q. Evaluate $\int_C \bar{z}\, dz$ from $z=0$ to $z=4+2i$ along the curve C given by (i) $z=t^2+it$,

(ii) the line from $z=0$ to $2i$ and then the line from $z=2i$ to $z=4+2i$

Sol^n: i) The given integral $\int_C (x-iy)(dx+idy)$

$[\bar{z}=x-iy]$ $= \int_C (x\,dx+y\,dy)+i\int_C (x\,dy-y\,dx)$

(25)

Here $x=4=t^2 \Rightarrow t=2$, $y=t$
as $z=4+2i$,

The parametric equations of C are $x=t^2$, $y=t$ from $t=0$ to $t=2$.

Then the line integral $= \int_0^2 (t^2 \cdot 2t\,dt + t\,dt) + i\int_0^2 (t^2 dt - t \cdot 2t\,dt)$

Note
$$\boxed{\begin{array}{l}(x-iy)(dx+idy)\\ =(t^2-it)(2t\,dt+i\,dt)\\ =(t^2 \cdot 2t+t)dt\\ \quad +i(t^2-2t^2)dt\end{array}}$$

$$= \left[\frac{t^4}{2}+\frac{t^2}{2}\right]_0^2 + i\left[\frac{t^3}{3}-\frac{2t^3}{3}\right]_0^2$$

$$= 10 - \frac{8i}{3}.$$

(ii) The line from $z=0$ to $z=2i$ is same as the line from $(0,0)$ to $(0,2)$ for which $x=0$ and the line integral equals

$$\int_{y=0}^{2} y\,dy = 2.$$

$dx=0$ and for $(x-iy)(dx+idy)$
$=(0-iy)(i\,dy)$
$=y\,dy$.

Again the line from $z=2i$ to $z=4+2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y=2$ and the line integral equals to for x ranges from 0 to 4.

$$\int_{x=0}^{4} x\,dx + i\int_{x=0}^{4} (-2)\,dx = 8 - 8i.$$

ie $(x-iy)(dx+idy)$
$\Rightarrow (x-2iy)(dx+0)$
$dy=0$
as $y=2$

A-2015
Q. Evaluate $\int_\Gamma \frac{1}{(z-1)^3}\,dz$, where Γ is the directed line segment from $z=1+i$ to $z=3+2i$ (2)

Sol^n: consider the path of integration joining the points $(1,1)$ and $(3,2)$ as a curve made of

(i) a line parallel to real axis from the point $(1,1)$ to the point $(3,1)$ and

(ii) a line parallel to the imaginary axis from the point $(3,1)$ to the point $(3,2)$.

See the figure for the line $(1,1)$ to $(3,2)$ divided by AB, BC

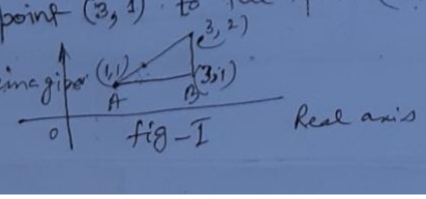


fig-I

(2a) from figure I

for AB

$z = x+iy$ (here $y=1$) so $dy=0$

For (i) we have $z = x+i$, $dy=0$ and x goes from 1 to 3.

Hence
$$\int_{(1,1)}^{(3,1)} \frac{dz}{(z-1)^3} = \int_1^3 \frac{dx}{(x+i-1)^3}$$
$$= \int_1^3 \frac{dx}{(x-1+i)^3}$$
$$= -\frac{1}{2}\left[\frac{1}{(x-1+i)^2}\right]_1^3$$
$$= -\frac{1}{2}\left[\frac{1}{(2+i)^2} - \frac{1}{i^2}\right]$$
$$= \frac{1}{2i^2} - \frac{1}{2(2+i)^2}$$
$$= -\frac{1}{2} - \frac{1}{2(3+4i)}$$
$$= -\frac{3+4i+1}{2(3+4i)}$$
$$= -\frac{4+4i}{2(3+4i)}$$
$$= -\frac{4(1+i)}{2(3+4i)}$$
$$= -\frac{2(1+i)}{3+4i}$$

$$-\frac{2(1+i)(3-4i)}{9-16i^2}$$
$$= -\frac{2}{25}(7-i)$$
$$= \frac{-14+2i}{25}$$

from figure I

for BC

For (ii) we have $z = 3+iy$, because $x=3$, $dx=0$ and y goes from 1 to 2.

Note

Here $dz = dx + i\,dy$

and $z-1 = 3+iy-1 = 2+iy$

and $dx = 0$

Therefore
$$\int_{(3,1)}^{(3,2)} \frac{dz}{(z-1)^3} = i\int_1^2 \frac{dy}{(2+iy)^3}$$
$$= i\left[\frac{1}{-2i(2+iy)^2}\right]_1^2$$
$$= \frac{1}{2}\left[\frac{1}{(2+i)^2} - \frac{1}{(2+2i)^2}\right]$$
$$= \frac{1}{2}\left[\frac{1}{3+4i} - \frac{1}{8i}\right]$$
$$= \frac{1}{2}\left[\frac{3-4i}{25} + \frac{i}{8}\right] = \frac{24-7i}{400}$$

Thus
$$\int_{1+i}^{3+2i} \frac{dz}{(z-1)^3} = -\frac{2(1+i)}{3+4i} + \frac{24-7i}{400} = -\frac{2(1+i)(3-4i)}{25} + \frac{24-7i}{400}$$
$$= \frac{-14+2i}{25} + \frac{24-7i}{400} = \frac{-8+i}{16}$$

[Simplify yourself]

2009

Q. Evaluate $\int_\Gamma \cos z \, dz$, where Γ is the directed line segment from $z = 2i$ to $z = -3$. (2)

Sol^n: Consider the path of integration joining the points $(0,2)$ and $(-3,0)$ as a curve made of

(i) a line parallel to the imaginary axis (imaginary axis itself) from the point $(0,2)$ to the point $(0,0)$. ~~and~~
on this line: $z = 0 + iy$, $dz = i\,dy$ and y goes 2 to 0.

(ii) a line parallel to the real axis from the point $(0,0)$ to $(-3,0)$. On this line: $z = x$, $dz = dx$, x goes [illegible]

Note
$z = x + iy$
here $y = 0$
$\therefore z = x$
$dz = dx$
$dy = 0$

Hence $$\int_\Gamma \cos z \, dz = \int_{(0,2)}^{(-3,0)} \cos z \, dz = \int_2^0 \cos(iy) \cdot i\,dy + \int_0^{-3} \cos x \cdot dx$$

$$= i\int_2^0 \cosh y \, dy + \int_0^{-3} \cos x \, dx \qquad [\cos(iy) = \cosh y]$$

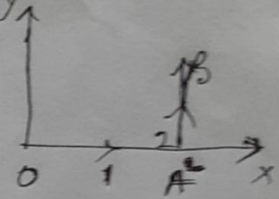
$$= i\left[\sinh y\right]_2^0 + \left[\sin x\right]_0^{-3}$$

$$= -i\sinh 2 - \sin 3$$

$$= -\sin 3 - i\sinh 2.$$

Complex Integration

Sum: Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the real axis from $z=0$ to $z=2$ and then along a line parallel to y-axis from $z=2$ to $z=2+i$



Solution: $\int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x-iy)^2 (dx+idy)$

$= \int_{OA} (x^2)\, dx$ because $y=0$, $dy=0$ i.e. $(x-0)^2 dx$; Along OA x varies from 0 to 2

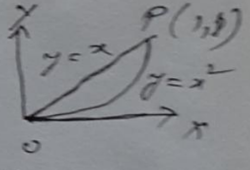
$+ \int_{AB} (2-iy)^2 i\,dy$ [because $x=2$, $dx=0$ in $(x-iy)(dx+idy)$; Along AB, y varies from 0 to 1

$= \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 dy = \left[\frac{x^3}{3}\right]_0^2 + i\int_0^1 (4-4iy-y^2)dy$

$= \frac{8}{3} + i\left[4y - 2iy^2 - \frac{y^3}{3}\right]_0^1 = \frac{8}{3} + i\left(4-2i-\frac{1}{3}\right)$

$= \frac{1}{3}(8+11i+6) = \frac{1}{3}(14+11i)$.

Evaluate $\int_0^{1+i} (x^2-iy)dz$ along a) $y=x$ b) along $y=x^2$



Sol^n: a) along $y=x \Rightarrow dy=dx$, $dz = dx + idx = (1+i)dx$

$\therefore \int_0^{1+i} (x^2-iy)dx = \int_0^1 (x^2-ix)(1+i)dx$ as x ranges from 0 to 1

$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1+i)\left(\frac{1}{3} - \frac{1}{2}i\right) = \frac{(1+i)(2-3i)}{6} = \frac{5}{6} - \frac{1}{6}i$

b) Along $y=x^2$, $dy = 2x\,dx$ so that $dz = dx + idy = (1+2ix)dx$; x varies from 0 to 1

$\therefore \int_0^{1+i} (x^2-iy)\,dx = \int_0^1 (x^2-ix^2)(1+2ix)dx = \int_0^1 x^2(1-i)(1+2ix)dx$

$= (1-i)\int_0^1 x^2(1+2ix)\,dx = (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1$

$= (1-i)\left[\frac{1}{3} + \frac{1}{2}i\right] = \frac{(1-i)(2+3i)}{6}$

$= \frac{1}{6}(2+3i-2i+3) = \frac{5}{6} + \frac{1}{6}i$

which is the required value of given integral.