Properties:-

i) $|z_1 + z_2| \le |z_1| + |z_2|$

$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

(ii) $|z_1 - z_2| \geq |z_1| \sim |z_2|$

(iii) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$|z_1 \cdot z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$

(iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Imaginary cube roots of unity.

Take, $x^3 = 1$

or, $x^3 - 1 = 0$

or, $(x-1)(x^2+x+1) = 0$

Then $x - 1 = 0$, $x^2 + x + 1 = 0$

or, $x = 1$, $x = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{1}{2}(-1 \pm i\sqrt{3})$

If we take, $\omega = \frac{1}{2}(-1 + i\sqrt{3})$

then $\omega^2 = \frac{1}{2}(-1 - i\sqrt{3})$

Thus the imaginary cube roots of unity are ω and ω^2.

⊛ Solved Example:-

Q1. Find the modulus and the amplitude of complex number

$$\frac{(1+i)(2+3i)}{(i-1)(2-3i)}$$

Sol:- Let $z = \frac{(1+i)(2+3i)}{(i-1)(2-3i)} = \frac{2+2i+3i+3i^2}{2i-2+3i-3i^2}$

$$= \frac{2+5i-3}{-2+5i+3} = \frac{-1+5i}{1+5i} = \frac{-(1-5i)^2}{(1+5i)(1-5i)}$$

$$= \frac{-(1-10i+25i^2)}{1-25i^2} = \frac{-1+10i+25}{1+25} = \frac{12}{13}+\frac{5}{13}i$$

Moreover $|z| = \sqrt{\left(\frac{12}{13}\right)^2+\left(\frac{5}{13}\right)^2} = \sqrt{\frac{144+25}{169}} = 1$

and $\operatorname{amp} z = \tan^{-1}\frac{5/13}{12/13} = \tan^{-1}\frac{5}{12}$.

Q2. Find the conjugate of the complex number $\frac{2-i}{(1-2i)^2}$

Sol: Let $z = \frac{2-i}{(1-2i)^2} = \frac{2-i}{1-4i+4i^2} = \frac{2-i}{1-4i-4}$

$$= -\frac{2-i}{3+4i} = -\frac{(2-i)(3-4i)}{(3+4i)(3-4i)} = -\frac{6-3i-8i-4}{9+16}$$

$$= -\frac{1}{25}(2-11i)$$

Hence, the conjugate of z is

$$-\frac{1}{25}(2+11i).$$

~~Q3. Find the conjugate of the~~

Q3. Simplify $(1-i)\left(1-\frac{1}{i}\right)$

Soln:- $(1-i)\left(1-\frac{1}{i}\right) = (1-i)\left(1+\frac{i^2}{i}\right)$

$= (1-i)(1+i) = 1-i^2 = 1+1 = 2.$

Q4: If $\sqrt[3]{x+iy} = a+ib$, then show that $\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$.

Soln:- We have $\sqrt[3]{x+iy} = a+ib$

or, $x+iy = (a+ib)^3$

or, $x+iy = a^3+3a^2(ib)+3a.(ib)^2+(ib)^3$

or, $x+iy = a^3+i3a^2b-3ab^2-ib^3$

~~$=(a^3-3ab^2)+i(3a^2$~~

or, $x+iy = (a^3-3ab^2)+i(3a^2b-b^3)$

Equating the real and imaginary parts from both sides, we get

$x = a^3-3ab^2$ and $y = 3a^2b-b^3$.

Hence, $\frac{x}{a} = a^2-3b^2$ and $\frac{y}{b} = 3a^2-b^2$

$\therefore \frac{x}{a}+\frac{y}{b} = a^2-3b^2+3a^2-b^2$

$= 4(a^2-b^2).$

Q5. If z be a complex number and $\frac{z+i}{z-i}$ be purely imaginary, then show that z lies on the circle whose centre is at the point $\frac{1}{2}(-1+i)$ and the radius is $\frac{1}{\sqrt{2}}$.

Sol^n:- Let $z = x+iy$

Then $\frac{z+1}{z-i} = \frac{(x+1)+iy}{x+i(y-1)}$

$= \frac{\{(x+1)+iy\}\{x-i(y-1)\}}{\{x+i(y-1)\}\{x-i(y-1)\}}$

$= \frac{x^2+x+ixy-i(xy+y-x-1)+y^2-y}{x^2+(y-1)^2}$

$= \frac{x^2+y^2+x-y}{x^2+(y-1)^2} + i\frac{x-y+1}{x^2+(y-1)^2}$

Since $\frac{z+1}{z-i}$ is purely imaginary,

therefore, $\frac{x^2+y^2+x-y}{x^2+(y-1)^2} = 0$

or, $x^2+y^2+x-y=0$

or, $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 = \frac{1}{4}+\frac{1}{4} = \left(\frac{1}{\sqrt{2}}\right)^2$

It is a equ^n of circle of radius $\frac{1}{\sqrt{2}}$ and the centre of the circle is at the point $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Therefore z lies on the circle whose centre is at the point $\frac{1}{2}(-1+i)$ and the radius is $\frac{1}{\sqrt{2}}$.

Q6. (a) Factorise: a^2-ab+b^2

(b) Show that $(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega) = x^3+y^3+z^3-3xyz$

where ω is an imaginary cube root of unity.

Sol^n:- (a) $a^2 - ab + b^2$

$= a^2 + (\omega + \omega^2)ab + b^2$ [Since $1+\omega+\omega^2=0$]

$= a^2 + \omega ab + \omega^2 ab + \omega^3 b^2$ [$\because \omega^3 = 1$]

$= a(a+\omega b) + \omega^2 b(a+\omega b)$

$= (a+\omega b)(a+\omega^2 b)$.

(b) $(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

$= (x+y+z)\{x^2+y^2\omega^3+z^2\omega^3+xy(\omega+\omega^2)$
$+ yz(\omega^4+\omega^2)+zx(\omega+\omega^2)\}$

$= (x+y+z)\{x^2+y^2+z^2-xy-yz-zx\}$

$= x^3+y^3+z^3-3xyz$. [Since $\omega^3=1$ and $\omega+\omega^2=-1$]

Q7. If ω be an imaginary cube root of unity, then prove that,

$$\frac{x+\omega y+\omega^2 z}{y+\omega z+\omega^2 x} = \omega$$

Sol^n:- $\dfrac{x+\omega y+\omega^2 z}{y+\omega z+\omega^2 x}$

$= \dfrac{\omega y+\omega^2 z+\omega^3 x}{y+\omega z+\omega^2 x}$ [$\because \omega^3=1$]

$= \dfrac{\omega(y+\omega z+\omega^2 x)}{(y+\omega z+\omega^2 x)}$

$= \omega$.