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De Moivre's theorem :-

Statement:-
For all integral values of n, the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$ and for all fractional values of n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$.

Proof:- Case-I: When n is a positive integer

we have $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$.

and $(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$= \cos 2\theta + i\sin 2\theta$.

$\therefore$ The theorem is true when $n=1$ and 2.

Let us assume that the theorem is true for a particular value m of n. (m being a positive integer).

Then we have,

$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

Multiplying both sides by $(\cos\theta + i\sin\theta)$ we get

$$\begin{aligned}(\cos\theta + i\sin\theta)^{m+1} &= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)\\ &= (\cos m\theta\cos\theta - \sin m\theta\sin\theta)\\ &\quad + i(\sin m\theta\cos\theta + \cos m\theta\sin\theta)\\ &= \cos(m+1)\theta + i\sin(m+1)\theta.\end{aligned}$$

Thus if the theorem be true for $n=m$, it is also true ~~too~~ for $n=m+1$.

Thus by Mathematical induction, the theorem is true for all positive integral values of n.

Case-II: When n is a negative integer.

Let $n = -m$, where m is a positive integer.

Then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)} \quad [\text{by Case-I}]$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta.$$

Thus the theorem is true for all negative integral values of n.

Case-III:- When n is fraction, positive or negative.

Let $n = \frac{p}{q}$, where q is a positive integer and p is any integer, positive or negative. Then by case-I,

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q\cdot\frac{\theta}{q} + i\sin q\cdot\frac{\theta}{q} = \cos\theta + i\sin\theta.$$

Extracting the q-th root of both sides, we see that one of the values of

$(\cos\theta + i\sin\theta)^{1/q}$ is $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)$.

Raising both sides to p-th power, we can say that one of the values of

$\{(\cos\theta + i\sin\theta)^{1/q}\}^p$ i.e., $(\cos\theta + i\sin\theta)^{p/q}$

i.e., $(\cos\theta + i\sin\theta)^n$ is

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

$$= \cos n\theta + i\sin n\theta$$

[by Case-I & Case-II]

Hence one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Thus the theorem holds good for all rational values of n.

This theorem is known as De Moivre's theorem.

Solved Example:-

Q1. Find the cube roots of (-1).

Solⁿ:- Let $x^3 = -1$

or, $x^3 = \cos\pi + i\sin\pi$

$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$, where $k = 0$ or any integer.

or, $x = \{\cos(2k+1)\pi + i\sin(2k+1)\pi\}^{1/3}$.

$= \cos\frac{1}{3}(2k+1)\pi + i\sin\frac{1}{3}(2k+1)\pi$, where $k = 0, 1, 2$.

Thus the required values are

$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, $\cos\pi + i\sin\pi$, $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

i.e., $\frac{1}{2}(1 + i\sqrt{3})$, -1, $\frac{1}{2}(1 - i\sqrt{3})$.

Q2. Find the values of $(1+i)^{1/5}$.

Soln:- We have, $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$

$= 2^{1/2}\left[\cos(2k\pi + \pi/4) + i\sin(2k\pi + \pi/4)\right]$

where $k = 0$, or any integer.

Hence,

$(1+i)^{1/5} = 2^{1/10}\left[\cos(2k\pi + \pi/4) + i\sin(2k\pi+\pi/4)\right]^{1/5}$

$= 2^{1/10}\left[\cos\left(2k+\frac{1}{4}\right)\frac{\pi}{5} + i\sin\left(2k+\frac{1}{4}\right)\frac{\pi}{5}\right]$

where $k = 0, 1, 2, 3$.

Q3 (i) If $z_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, then prove that $z_1 z_2 z_3 \cdots$ to $\infty = -1$.

(ii) Find the general value of the real angle θ which satisfies the equn $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\cdots(\cos n\theta + i\sin n\theta) = 1$.

Soln:- (i) $z_1 z_2 z_3 \cdots$ to $\infty = \lim_{n\to\infty}(z_1 z_2 \cdots z_n)$

$= \lim_{n\to\infty}\left(\cos \pi/2 + i\sin \pi/2\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\cdots\left(\cos\frac{\pi}{2^n} + i\sin\frac{\pi}{2^n}\right)$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots \text{to } \infty\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots \text{to } \infty\right)$

$= \cos\left(\frac{\pi/2}{1 - 1/2}\right) + i\sin\frac{\pi/2}{1-1/2} = \cos\pi + i\sin\pi = -1.$

(ii) $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\cdots(\cos n\theta + i\sin n\theta) = 1.$

or, $\cos(\theta + 2\theta + \cdots + n\theta) + i\sin(\theta + 2\theta + \cdots + n\theta) = \cos 0 + i\sin 0.$

or, $\cos\frac{n(n+1)\theta}{2} + i\sin\frac{n(n+1)\theta}{2} = \cos 2k\pi + i\sin 2k\pi$

where $k = 0$, or any integer.

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or, $\frac{1}{2}n(n+1)\theta = 2k\pi$

or, $\theta = \frac{4k\pi}{n(n+1)}$, where $k=0$, or any integer.

Q4. If $x+\frac{1}{x} = 2\cos\frac{\pi}{7}$, then show that

$$x^7+\frac{1}{x^7} = -2.$$

Solⁿ: We have, $x+\frac{1}{x} = 2\cos\frac{\pi}{7}$

or. $x^2+1 = 2x\cos\pi/7$

~~or x^2-2x~~

or, $x^2-2x\cos\pi/7+1=0.$

or, $x = \frac{2\cos\pi/7 \pm \sqrt{4\cos^2\frac{\pi}{7}-4}}{2}$

$= \frac{2(\cos\pi/7 \pm i\sin\pi/7)}{2}$

$= \cos\pi/7 \pm i\sin\pi/7$

Therefore $x^7 = (\cos\pi/7 \pm i\sin\pi/7)^7$

$= \cos\pi \pm i\sin\pi$

$= -1$

Hence $x^7+\frac{1}{x^7} = -1+\frac{1}{-1} = -2.$

Solve the following problem.

Q1. Find the values of $(1-i)^7$.

Q2. Solve $x^5=1$.

Q3. If n be a positive integer, then prove that $(1+i)^n+(1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$.