

~~Saturday~~

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		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

De Moivre's theorem :-Statement:-

For all integral values of  $n$ , the values of  $(\cos \theta + i \sin \theta)^n$  is  $(\cos n\theta + i \sin n\theta)$  and for all fractional values of  $n$ , one of the values of  $(\cos \theta + i \sin \theta)^n$  is  $(\cos n\theta + i \sin n\theta)$ .

Case-I:

Proof:- When  $n$  is a positive integer we have  $(\cos \theta + i \sin \theta)^1 = \cos \theta + i \sin \theta$  and  $(\cos \theta + i \sin \theta)^2 = \cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta = \cos 2\theta + i \sin 2\theta$ .

$\therefore$  The theorem is true when  $n=1$  and  $2$ . Let us assume that the theorem is true for a particular value  $m$  of  $n$ . ( $m$  being a positive integer).

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then we have,

$$(\cos \theta + i \sin \theta)^m = \cos m\theta + i \sin m\theta$$

Multiplying both sides by  $(\cos \theta + i \sin \theta)$  we get

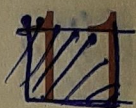
$$\begin{aligned} (\cos \theta + i \sin \theta)^{m+1} &= (\cos m\theta + i \sin m\theta) (\cos \theta + i \sin \theta) \\ &= (\cos m\theta \cos \theta - \sin m\theta \sin \theta) \\ &\quad + i(\sin m\theta \cos \theta + \cos m\theta \sin \theta) \\ &= \cos(m+1)\theta + i \sin(m+1)\theta \end{aligned}$$

Thus if the theorem be true for  $n=m$ , it is also true for  $n=m+1$ .

Thus by mathematical induction, the theorem is true for all positive integral values of  $n$ .



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Case-II: When  $n$  is a negative integer.

Let  $n = -m$ , where  $m$  is a positive integer.

$$\begin{aligned} \text{then } (\cos \theta + i \sin \theta)^n &= (\cos \theta + i \sin \theta)^{-m} \\ &= \frac{1}{(\cos \theta + i \sin \theta)^m} \\ &= \frac{1}{(\cos m\theta + i \sin m\theta)} \quad [\text{by Case I}] \\ &= \frac{1}{(\cos m\theta - i \sin m\theta)} \\ &= \frac{(\cos m\theta - i \sin m\theta)}{(\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)} \\ &= \frac{(\cos m\theta - i \sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} \\ &= \cos m\theta - i \sin m\theta \\ &= \cos(-m)\theta + i \sin(-m)\theta \\ &= \cos n\theta + i \sin n\theta. \end{aligned}$$

Thus the theorem is true for all negative integral values of  $n$ .

~~Tuesday~~

Case-III:- When  $n$  is fraction, positive or negative.

Let  $n = \frac{p}{q}$ , where  $q$  is a positive integer and  $p$  is any integer, positive or negative.

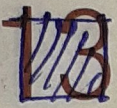
Then by Case-I,

$$\left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right)^q = \cos q \cdot \frac{\theta}{q} + i \sin q \cdot \frac{\theta}{q} = \cos \theta + i \sin \theta.$$

Extracting the  $q$ -th root of both sides, we see that one of the values of

$$\left( \cos \theta + i \sin \theta \right)^{1/q} \text{ is } \left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right).$$





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Raising both sides to  $p$ -th power, we can say that one of the values of

$$\left\{ (\cos \theta + i \sin \theta)^{1/2} \right\}^p \text{ i.e., } (\cos \theta + i \sin \theta)^{p/2}$$

i.e.,  $(\cos \theta + i \sin \theta)^n$  is

$$(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})^p = \cos \frac{p}{2} \theta + i \sin \frac{p}{2} \theta$$

$$= \cos n\theta + i \sin n\theta$$

[by Case-I & Case-II]

Hence one of the values of  $(\cos \theta + i \sin \theta)^n$  is  $\cos n\theta + i \sin n\theta$ .

Thus the theorem holds good for all rational values of  $n$ .

14 ~~Thursday~~ This theorem is known as De Moivre's theorem.

Solved Example:-

Q1. Find the cube roots of  $(-1)$ .

Sol:- Let  $x^3 = -1$

$$\text{or, } x^3 = \cos \pi + i \sin \pi$$

$$= \cos (2k\pi + \pi) + i \sin (2k\pi + \pi)$$

where  $k \geq 0$  or any integer.

$$\text{or, } x = \left\{ \cos (2k+1)\pi + i \sin (2k+1)\pi \right\}^{1/3}$$

$$= \cos \frac{1}{3}(2k+1)\pi + i \sin \frac{1}{3}(2k+1)\pi, \text{ where } k=0, 1, 2.$$

Thus the required values are  $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ ,  $\cos \pi + i \sin \pi$ ,  $\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$

$$\text{i.e., } \frac{1}{2}(1 + i\sqrt{3}), -1, \frac{1}{2}(1 - i\sqrt{3})$$



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Q2. Find the values of  $(1+i)^{1/5}$ .

Sol<sup>n</sup>:- we have,  $1+i = \sqrt{2} \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$   
 $= \sqrt{2} \left( \cos \pi/4 + i \sin \pi/4 \right)$   
 $= 2^{1/2} \left[ \cos(2k\pi + \pi/4) + i \sin(2k\pi + \pi/4) \right]$   
 where  $k=0$ , or any integer.

Hence,  
 $(1+i)^{1/5} = 2^{1/10} \left[ \cos(2k\pi + \pi/4) + i \sin(2k\pi + \pi/4) \right]^{1/5}$   
 $= 2^{1/10} \left[ \cos \left( 2k + \frac{1}{4} \right) \frac{\pi}{5} + i \sin \left( 2k + \frac{1}{4} \right) \frac{\pi}{5} \right]$   
 where  $k=0, 1, 2, 3$ .

Q3 (i) If  $z_1 = \cos \frac{\pi}{2^n} + i \sin \frac{\pi}{2^n}$ , then prove that  
 $z_1 z_2 z_3 \dots \text{to } \infty = -1$ .

(ii) Find the general value of the ~~Saturday~~ 16  
 real angle  $\theta$  which satisfies the eqn  
 $(\cos \theta + i \sin \theta) (\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$ .

Sol<sup>n</sup>:- (i)  $z_1 z_2 z_3 \dots \text{to } \infty = \lim_{n \rightarrow \infty} (z_1 z_2 \dots z_n)$   
 $= \lim_{n \rightarrow \infty} \left( \cos \pi/2 + i \sin \pi/2 \right) \left( \cos \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2} \right) \dots \left( \cos \frac{\pi}{2^n} + i \sin \frac{\pi}{2^n} \right)$   
 $= \cos \left( \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots \text{to } \infty \right) + i \sin \left( \frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots \text{to } \infty \right)$   
 $= \cos \left( \frac{\pi/2}{1-1/2} \right) + i \sin \frac{\pi/2}{1-1/2} = \cos \pi + i \sin \pi = -1$ .

(ii)  $(\cos \theta + i \sin \theta) (\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$   
 $\therefore \cos(\theta + 2\theta + \dots + n\theta) + i \sin(\theta + 2\theta + \dots + n\theta) = \cos 0 + i \sin 0$   
 $\therefore \cos \frac{n(n+1)\theta}{2} + i \sin \frac{n(n+1)\theta}{2} = \cos 2k\pi + i \sin 2k\pi$   
 where  $k=0$ , or any integer.



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$$or, \frac{1}{2} n(n+1) \theta = 2k\pi$$

$$or, \theta = \frac{4k\pi}{n(n+1)}, \text{ where } k=0, \text{ or any integer.}$$

Q4. If  $x + \frac{1}{x} = 2 \cos \frac{\pi}{7}$ , then show that

$$x^7 + \frac{1}{x^7} = -2.$$

Sol<sup>n</sup>: we have,  $x + \frac{1}{x} = 2 \cos \frac{\pi}{7}$

$$or, x^2 + 1 = 2x \cos \frac{\pi}{7}$$

$$or, x^2 - 2x \cos \frac{\pi}{7} + 1 = 0$$

$$or, x^2 - 2x \cos \frac{\pi}{7} + 1 = 0$$

$$or, x = \frac{2 \cos \frac{\pi}{7} \pm \sqrt{4 \cos^2 \frac{\pi}{7} - 4}}{2}$$

$$= \frac{2 (\cos \frac{\pi}{7} \pm i \sin \frac{\pi}{7})}{2}$$

$$= \cos \frac{\pi}{7} \pm i \sin \frac{\pi}{7}$$

therefore  $x^7 = (\cos \frac{\pi}{7} \pm i \sin \frac{\pi}{7})^7$

$$= \cos \pi \pm i \sin \pi$$

$$= -1$$

Hence  $x^7 + \frac{1}{x^7} = -1 + \frac{1}{-1} = -2.$

Q5. Solve the following problem.

Q1. Find the values of  $(1-i)^7$ .

Q2. Solve  $x^5 = 1$ .

Q3. If  $n$  be a positive integer, then prove that  $(1+i)^n + (1-i)^n = 2^{\frac{n+1}{2}} \cos \frac{n\pi}{4}$ .