

Learning Materials for Sem-2.
unit - I , CC- 4

Topic - Differential Equation
(Variation of Parameters)

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① solve by the method of variation of parameters: $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

$$\frac{d^2y}{dx^2} + a^2y = \sec ax \quad \text{①}$$

$$\text{let us first solve } \frac{d^2y}{dx^2} + a^2y = 0 \quad \text{②}$$

let $y = e^{mx}$ be a trial solution of ②. Then the A.E of ② is →

$$e^{mx} (m^2 + a^2) = 0$$

$$(m^2 + a^2) = 0 \quad [\text{Since } e^{mx} \neq 0]$$

$$m = \pm ai$$

Then $y_c = C_1 F$ of ①

$$= C_1 \cos ax + C_2 \sin ax \quad [C_1, C_2 \text{ are arbitrary constant}]$$

let $y_1 = \cos ax$, $y_2 = \sin ax$.

Here $w(y_1, y_2)$ = wronskian of y_1, y_2

$$= \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = \cos ax \left(\frac{1}{a} \right) - \sin ax (\beta + \beta) + \sin ax (\alpha + \beta) =$$

$$= \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0$$

Therefore y_1 and y_2 are linearly independent.

let y_p be a particular solution of ①.

let $y_p = u(x) \cos ax + v(x) \sin ax$.

$$\frac{dy_p}{dx} = \frac{du}{dx} \cos ax + \frac{dv}{dx} \sin ax - ua \sin ax + va \cos ax$$

let us choose u and v in such a manner that

$$\frac{du}{dx} \cos ax + \frac{dv}{dx} \sin ax = 0 \quad \text{③}$$

therefore

$$\frac{dy_p}{dx} = va \cos ax - ua \sin ax$$

$$\frac{d^2y_p}{dx^2} = -a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - a^2 u \cos ax - a^2 v \sin ax$$

since y_p is a particular solution of ①, therefore →

$$\frac{d^2y_p}{dx^2} + a^2 y_p = \sec ax$$

$$\text{or, } -a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - a^2 u \cos ax - a^2 v \sin ax = \sec ax$$

$$\text{or, } a \cos ax \frac{dv}{dx} - a \sin ax \frac{du}{dx} = \sec ax$$

$$\text{or, } -a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - \sec ax = 0 \quad \text{④}$$

From ③ and ④ we get →

$$-a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - \sec ax = 0$$

$$\cos ax \frac{du}{dx} + \sin ax \frac{dv}{dx} = 0$$

$$\text{Therefore } \frac{dy_p}{dx} = -e^{-x}u + (1-x)e^{-x}v$$

$$\frac{d^2y_p}{dx^2} = e^{-x}u + (x-2)e^{-x}v - e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx}$$

Since y_p is a particular solution of ①.

$$\text{Therefore } \frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + y_p = \frac{e^{-x}}{x}$$

$$e^{-x}u + (x-2)e^{-x}v - e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} - 2e^{-x}u + 2(1-x)e^{-x}v + ue^{-x} + vxe^{-x} = \frac{e^{-x}}{x}$$

$$-e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} = \frac{e^{-x}}{x} \quad \text{--- ②}$$

From ① and ② we get →

$$-e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} - \frac{e^{-x}}{x} = 0$$

$$e^{-x}\frac{du}{dx} + xe^{-x}\frac{dv}{dx} + 0 = 0$$

$$\begin{aligned} \frac{du}{dx} &= -\frac{dv/dx}{e^{-2x}} &= \frac{1}{-e^{-2x} \cdot 2 - e^{-2x}(1-2)} \\ &= \frac{1}{-e^{-2x}[x+1-2]} \\ &= \frac{1}{-e^{-2x}x} \end{aligned}$$

$$\frac{du}{dx} = -1 \quad \text{--- ③} \quad \frac{dv}{dx} = \frac{1}{x} \quad \text{--- ④}$$

$$\text{Integrating ③ we get } u = -x + c_3$$

$$\text{Integrating ④ we get } v = \log|x| + c_4 \quad [c_3, c_4 \text{ are arbitrary constant}]$$

$$\text{Therefore } y_p = (-x+c_3)e^{-x} + (\log|x|+c_4)x e^{-x}$$

Then the required solution is →

$$\begin{aligned} y &= y_p + y_c \\ &= (c_1 + c_2 x)e^{-x} + (-x+c_3)e^{-x} + (\log|x|+c_4)x e^{-x} \\ &= c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{-x} - x e^{-x} + \log|x| x e^{-x} + c_4 x e^{-x} \\ &= (c_1 + c_3)e^{-x} + (c_2 + c_4)x e^{-x} - x e^{-x} + \log|x| x e^{-x} \\ &= c_5 e^{-x} + c_6 x e^{-x} - x e^{-x} + \log|x| x e^{-x}. \end{aligned}$$

③ solve by the method of variation of parameters:

$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \log x, x > 0$, it is being given that $y=x$ and $y=\frac{1}{x}$ are two linearly independent solution of it reduced equation.

Since $y=x$ and $y=\frac{1}{x}$ are two linearly independent solution of ①. therefore

$$y_c = c_1 x + c_2 \frac{1}{x} \quad [\text{where } c_1, c_2 \text{ are arbitrary constant}]$$

Here $y_1 = x$ and $y_2 = \frac{1}{x}$ are linearly independent solution.

Let y_p be a particular solution of ①.

$$\text{Then } y_p = ux + \frac{v}{x} \quad [\text{where } u \text{ and } v \text{ are function of } x \text{ or constant}]$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot x + u + \frac{dv}{dx} \cdot \frac{1}{x^2} - \frac{1}{x^2} v$$

Let us choose u and v in such a manner that

$$\frac{du}{dx} \cdot x + \frac{dv}{dx} \cdot \frac{1}{x^2} = 0 \quad \text{--- (1)}$$

Therefore, $\frac{dy}{dx} = u - \frac{1}{x^2} v$

$$\frac{d^2y}{dx^2} = \frac{du}{dx} - \frac{1}{x^2} \frac{dv}{dx} + \frac{2}{x^3} v$$

Since y_p is a particular solution of (1). therefore

$$\frac{d^2y_p}{dx^2} + \frac{1}{x} \frac{dy_p}{dx} - \frac{1}{x^2} y_p = \log x$$

$$\frac{du}{dx} - \frac{1}{x^2} \frac{dv}{dx} + \frac{2}{x^3} v + \frac{1}{x} u - \frac{1}{x^3} v - \frac{u}{x^2} = \log x$$

$$\frac{du}{dx} - \frac{1}{x^2} \frac{dv}{dx} = \log x \quad \text{--- (2)}$$

From (1) and (2) we get,

$$\frac{du}{dx} - \frac{1}{x^2} \frac{dv}{dx} - \log x = 0$$

$$x \frac{du}{dx} + \frac{1}{x} \frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{\log x} = \frac{\frac{dv}{dx}}{-x \log x} = \frac{1/x + x}{x^2 + \frac{1}{x} \log x} = \frac{1/x + x}{x^2 + \frac{1}{x} \log x}$$

$$\frac{du}{dx} = \frac{\log x}{2} \quad \text{--- (3)} \quad \frac{dv}{dx} = -\frac{1}{2} x^2 \log x \quad \text{--- (4)}$$

Integrating (3) we get, $u = \left[\frac{1}{2} x^2 + c_3 \right] \cdot x^2 \cdot (\log x + 1)$

$$\text{Integrating (4) we get, } v = -\frac{1}{6} (\log x - 1/3) x^3 + c_4$$

$$\text{Therefore } y_p = \left[\frac{1}{2} x^2 + c_3 \right] x^2 + \left[-\frac{1}{6} (\log x - 1/3) x^3 + c_4 \right] \frac{1}{x} + c_5 \\ = \frac{1}{2} x^2 - \frac{1}{6} (\log x - 1/3) x^2 + c_3 x + c_4 x + c_5 x^{-1} e^{c_6}$$

Then the required general solution is \rightarrow substitute to both (2) and (3) we get,

$$\begin{aligned} y &= y_p + y_c \\ &= c_2 x + c_3 x^2 + \frac{1}{2} x^2 - \frac{1}{6} (\log x - 1/3) x^2 + \left(c_4 + c_5 \right) x + \frac{c_6}{x} \\ &= \frac{1}{2} x^2 - \frac{1}{6} (\log x - 1/3) x^2 + \left(c_4 + c_5 \right) x + \frac{c_6}{x} \\ &= \frac{1}{2} x^2 - \frac{1}{6} (\log x - 1/3) x^2 + c_4 x + c_5 \frac{1}{x} + c_6 x^{-1} \end{aligned}$$

(4) solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + \gamma = \cos 2x \quad \text{--- (2013)}$$

$$\frac{d^2y}{dx^2} + \gamma = \cos 3x \quad \text{--- (1)}$$

Let us first solve $\frac{d^2y}{dx^2} + \gamma = 0$ --- (2)

Let $y = e^{mx}$ be a trial solution of (2). Then A.F of (2) is $m^2 + \gamma$

$$e^{mx} (m^2 + \gamma) = 0 \quad [\text{Since } e^{mx} \neq 0]$$

$$(m^2 + \gamma) = 0 \quad [\text{Since } e^{mx} \neq 0]$$

$$m^2 = -\gamma \quad \text{or} \quad m = \pm i$$

Therefore $y_c = C_1 F$ of (1)

$$= C_1 \cos 3x + C_2 \sin 3x \quad [C_1, C_2 \text{ are arbitrary constants}]$$

$$\text{Let } y_1 = \cos 3x \quad \text{and} \quad y_2 = \sin 3x$$

Here $\omega(y_1, y_2) = \text{constant of } y_1, y_2$

$$\begin{aligned} \omega(y_1, y_2) &= \left| \begin{array}{cc} \cos 3x & \sin 3x \\ -\sin 3x & \cos 3x \end{array} \right| = \cos^2 3x + \sin^2 3x \\ &= 1 \neq 0 \end{aligned}$$

Therefore y_1 and y_2 are linearly independent. So $\omega(y_1, y_2) = 1 \neq 0$

Let y_p be a particular solution of (1).

$$y_p = u \sin 3x + v \cos 3x \quad [\text{Here } u \text{ and } v \text{ are functions of } x]$$

$$\frac{dy_p}{dx} = \sin 3x \frac{du}{dx} + \cos 3x \frac{dv}{dx} + u \cos 3x - v \sin 3x$$

$$\frac{d^2y_p}{dx^2} = \frac{d}{dx} \left(\sin 3x \frac{du}{dx} + \cos 3x \frac{dv}{dx} + u \cos 3x - v \sin 3x \right) = \sin 3x \frac{d^2u}{dx^2} + \cos 3x \frac{d^2v}{dx^2} + \cos 3x \frac{du}{dx} - \sin 3x \frac{dv}{dx}$$

$$\text{Let us choose } u \text{ and } v \text{ in such a manner that} \quad \frac{d^2u}{dx^2} + \cos 3x \frac{du}{dx} = 0 \quad \text{--- (I)}$$

$$\frac{d^2v}{dx^2} + \cos 3x \frac{dv}{dx} = 0 \quad \text{--- (II)}$$

$$\text{Then } \frac{dy_p}{dx} = u \cos 3x - v \sin 3x \quad \text{and} \quad \frac{d^2y_p}{dx^2} = \frac{du}{dx} \cos 3x - \frac{dv}{dx} \sin 3x = u \sin 3x - v \cos 3x.$$

$$\frac{d^2y_p}{dx^2} + \gamma y_p = \cos 3x \quad \text{--- (III)}$$

$$\text{or, } \frac{d^2y_p}{dx^2} + \gamma y_p = \cos 3x - \sin 3x = \cos 3x + \gamma \sin 3x$$

From (1) and (III) we get \rightarrow $\sin 3x + \gamma \cos 3x$ is a particular solution of (1). Therefore $y_p = \sin 3x + \gamma \cos 3x$.

$$\begin{aligned} \frac{d^2y_p}{dx^2} + \gamma y_p &= \cos 3x \\ \cos 3x \frac{du}{dx} - \frac{dv}{dx} \sin 3x - (\sin 3x + \gamma \cos 3x) + (\sin 3x + \gamma \cos 3x) &= \cos 3x \end{aligned}$$

$$\text{or, } \cos 3x \frac{du}{dx} - \frac{dv}{dx} \sin 3x = \cos 3x \quad \text{--- (IV)}$$

$$\text{From (1) and (IV) we get } \rightarrow \sin 3x + \gamma \cos 3x \text{ is a particular solution of (1).}$$

$$\sin 3x + \gamma \cos 3x \frac{dv}{dx} = 0$$

$$\frac{\sin 3x}{\cos 3x} + \frac{\gamma \cos 3x}{\cos 3x} \frac{dv}{dx} = 0 \quad \text{or} \quad \tan 3x + \gamma = 0$$

$$\frac{dv}{dx} = -\frac{1}{\tan 3x} = -\frac{1}{\frac{\sin 3x}{\cos 3x}} = -\frac{\cos 3x}{\sin 3x}$$

$$\frac{du}{dx} = e^{2x} \quad \text{--- (2)}$$

Integrating (2) we get $u = \log(\sin x) + C_3 = V + \frac{C_3}{\sin x}$ where $V = \int e^{2x} dx$

Integrating (1) we get $V = Lx + C_4$ --- (3) a particular solution of (1)

Therefore $y_p = (\log(\sin x) + C_3) \sin x + (-x + C_4) \cos x$
Then the required general solution is $y = y_p + y_c$

$$= 4 \cos x [e_2 \sin x + \log(\sin x) + C_3]$$

$$= 4 \cos x + e_4 \cos x + e_2 \sin x + e_3 \sin x + [\log(\sin x)] \sin x - x \cos x$$

$$= e_5 x (e_1 + e_4) + \sin x (e_2 + e_3) + \log(\sin x) \sin x - x \cos x$$

$$= e_5 \cos x + e_6 \sin x + [\log(\sin x)] \sin x - x \cos x.$$

5. solve by the method of variation of parameters $x^2 \frac{dy}{dx^2} - x(x+2) \frac{dy}{dx} - x(x+2)y = 0$
given that $y=x$ and $y=x^2$ are independent solution of the second
equation.

$$x^2 \frac{d^2y}{dx^2} - x(x+2) \frac{dy}{dx} + (x+2)y = x^3$$

$$\frac{d^2y}{dx^2} - \left(\frac{x+2}{x}\right) \frac{dy}{dx} + \left(\frac{x+2}{x^2}\right)y = x \quad \text{--- (1)}$$

Since $y=x$ and $y=x^2$ are two linearly independent solution of (1).

Therefore $\rightarrow y_c = e_1 x + e_2 x^2$ [where e_1 and e_2 are arbitrary constant]

Let y_p be a particular solution of (1)

Then $y_p = ux + vx^2$ [where u and v are function of x or constant]

$$\frac{dy_p}{dx} = \frac{du}{dx} \cdot x + \frac{dv}{dx} x^2 + u + v(x+1)x^2$$

Let us choose u and v in such a manner that $\rightarrow u \cos x \frac{du}{dx} = -\frac{v}{x^2}$

$$\frac{du}{dx} x + \frac{dv}{dx} x^2 = 0 \quad \text{--- (2)}$$

$$\therefore \frac{dy_p}{dx} = u + v(x+1)x^2$$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx} + \frac{dv}{dx} (x+1)x^2 + v x^2 (x+2)$$

$$\frac{d^2y_p}{dx^2} - \left(\frac{x+2}{x}\right) \frac{dy_p}{dx} + \left(\frac{x+2}{x^2}\right)y_p = x$$

$$\frac{du}{dx} + \frac{dv}{dx} (x+1)x^2 = x \quad \text{--- (3)}$$

$$\text{From (2) and (3) we get } \frac{du}{dx} = \frac{v}{x^2} \text{ and } \frac{dv}{dx} = -\frac{u}{x^3}$$

$$\frac{du}{dx} = \frac{1}{x^2} \frac{v}{x^2} = \frac{1}{x^4} v = \frac{1}{x^4} \frac{u}{x^3}$$

$\frac{du}{dx} = -1 \quad \text{--- (3)}$
 Integrating (3) we get $u = -x + C_3$
 Integrating (4) we get $v = -e^{-x} + C_4$
 therefore
 $y_p = -x \cdot x - e^{-x} \cdot x^2 + C_3 + C_4$
 $= -x^2 - x + C_3$
 $= -(x^2 + x) + C_3$

Therefore the required general solution is \rightarrow
 $y = y_p + y_c$
 $= -x^2 - x + C_3 + C_2 x e^{-x} + C_5$
 $= -x^2 - x + C_6 x + C_7 x e^{-x}$

6. Solve by the method of variation of parameters $(1+x) \frac{dy}{dx} + x \frac{dy}{dx} - y = (1+x)^2$
 It is given that $y=x$ and $y=e^{-x}$ are independent solutions of the reduced equation.
 $(1+x) \frac{dy}{dx} + x \frac{dy}{dx} - y = (1+x)^2$ of homogeneous part of
 $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$ has a particular solution y_p such that y_p is also a solution of
 $\frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = (1+x)^2$ \rightarrow $\frac{dy}{dx} + (1+x)y = (1+x)^2$
 Since $y=x$ and $y=e^{-x}$ are two linearly independent solutions of (3).
 Therefore $y_1 = C_1 x + C_2 e^{-x}$ [C_1 and C_2 are A.C.]
 Let y_p be a particular solution of (3).
 Then $y_p = ux + ve^{-x}$ [where u and v are function of x or constant].
 $\frac{dy_p}{dx} = \frac{du}{dx} \cdot x + u + v \frac{d}{dx}(e^{-x})$
 Let us choose u and v in such a manner that \rightarrow
 $\frac{du}{dx} \cdot x + \frac{d}{dx}(e^{-x}) = 0 \quad \text{--- (4)}$
 therefore $\frac{du}{dx} = u - e^{-x}v$

$\frac{d^2y_p}{dx^2} = \frac{du}{dx} - e^{-x} \frac{dv}{dx} + e^{-x}v$
 Since y_p be a particular solution of (3), therefore \rightarrow
 $\frac{d^2y_p}{dx^2} + x \frac{dy_p}{dx} - \frac{1}{1+x} y_p = 1+x$ which is a solution of (3) of
 $\frac{du}{dx} - e^{-x} \frac{dv}{dx} = 1+x \frac{(1+x)^2}{(1+x)} - \frac{(1+x)^2}{(1+x)} + \frac{1+x}{(1+x)} = 0$
 From (3) and (4) we get \rightarrow
 $\frac{du}{dx} - e^{-x} \frac{dv}{dx} - (1+x) = 0$
 $\frac{du}{dx} + e^{-x} \frac{dv}{dx} + 1 = 0$

$$\frac{du}{dx} = \frac{\frac{dy}{dx}}{e^{-x}(1+x)} = \frac{1}{-x(1+x)} = \frac{1}{e^{-x} + e^{-x} \cdot x} = \frac{1}{e^{-x} + e^x - 1}$$

Integrating ③ we get $u = x$

Integrating ④ we get $v = e^x(1-x)$
therefore $y_p = x^2 + (1-x) e^x \cdot e^x$

$$= x^2 + 1 - x$$

Therefore the required general solution is $y = y_p + y_c$
Solve by the method of variation of parameters $(2x+1)(1+x) \frac{dy}{dx} + 2x \frac{dy}{dx} - 2y = (2x+1)^2$
It being given that $\gamma_1 = x$ and $\gamma_2 = \frac{1}{1+x}$ are two linearly independent solution of the corresponding homogeneous equation.

$$(2x+1)(1+x) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = (2x+1)^2$$

$$\frac{d^2y}{dx^2} + \frac{2x}{(2x+1)(1+x)} \frac{dy}{dx} - \frac{(2x+1)}{(1+x)} y = 0 \quad \text{--- ①}$$

Since $\gamma_1 = x$ and $\gamma_2 = \frac{1}{1+x}$ are two linearly independent solution of ①. Therefore $\gamma_1 = c_1 x + c_2 \frac{1}{1+x}$ [c_1 and c_2 are arbitrary constant]

Let y_p be a particular solution of ① \Rightarrow $\frac{dy_p}{dx} = A + \frac{B}{1+x}$
Then $y_p = ux + \frac{v}{1+x}$ [where u and v are function of x or constant]

$$\frac{du}{dx} + \frac{dv}{dx} \cdot \frac{1}{1+x} = 0 \quad \text{--- ②}$$

$$\text{Therefore } \frac{dy_p}{dx} = u + \frac{v}{-(x+1)^2}$$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx} - \frac{dv}{dx} \left(\frac{1}{x+1} \right)^2 + \frac{1}{(x+1)^3} v$$

$$\text{Since } y_p \text{ is a particular solution, therefore } \frac{d^2y_p}{dx^2} + \frac{2x}{(2x+1)(x+1)} \frac{dy_p}{dx} - \frac{2}{(2x+1)(x+1)} y_p = 0 \quad \text{--- ③}$$

$$\frac{du}{dx} - \frac{1}{(x+1)^2} \frac{dv}{dx} = \frac{2x+1}{(x+1)^2} \quad \text{--- ④}$$

$$\text{From ③ and ④ we get } \frac{du}{dx} = \frac{v}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{v}{(x+1)^2} \quad \text{--- ⑤}$$

From ② and ⑤ we get

$$\begin{aligned} \frac{du}{dx} &= -\frac{1}{(x+1)^2} \cdot \frac{dy}{dx} = 0 \\ \lambda \frac{du}{dx} + \frac{1}{(x+1)} \frac{dy}{dx} + 0 &= 0 \quad \text{[since } \lambda \neq 0 \text{]} \\ \lambda \frac{du}{dx} &= -\frac{1}{(x+1)} \frac{dy}{dx} \quad \text{[cancel } \lambda \text{]} \\ \frac{du}{dx} &= -\frac{1}{\lambda(x+1)} \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} &= \frac{(x+1)^2}{\lambda+1+\lambda} \cdot \frac{dy}{dx} \quad \text{[cancel } \lambda \text{]} \\ &= \frac{(x+1)^2}{2x+1} \cdot \frac{dy}{dx} \quad \text{[cancel } \lambda \text{]} \end{aligned}$$

$$\frac{du}{dx} = 1 \quad \text{--- (3)}$$

Integrating (3) we get $u = x$

$$\text{Integrating (3) we get } v = \frac{1}{6}(2x^2 + 3x^2)$$

$$\text{Therefore } y_p = \frac{x^2}{6} \cdot \frac{4x+3}{x+1}$$

Then the required general solution is $y = y_p + y_c$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x^2}{6} \cdot \frac{4x+3}{x+1} + q_1 x + \frac{c_2}{x+1} \\ &= \frac{2x^2}{6} \cdot \frac{4x+3}{x+1} + q_1 x + \frac{c_2}{x+1} \end{aligned}$$

$$\text{⑧ Solve by variation of parameters: } \frac{dy}{dx} + 2 \frac{dy}{dx} + y = \frac{c_2}{x+1}$$

$$\text{Let us first solve } \frac{dy}{dx} + 2 \frac{dy}{dx} + y = 0 \quad \text{--- (4)}$$

Let $y = e^{mx}$ be a trial solution of (4). Then the A.E of (4) is $m^2 + 2m + 1 = 0$ [Since $e^{mx} \neq 0$]

$$(m+1)^2 = 0$$

$$m = -1, -1$$

Therefore $y_c = (c_1 + c_2 x)e^{-x}$ [c_1, c_2 are arbitrary constants]

$$\text{Let } y_1 = e^{-x} \text{ and } y_2 = xe^{-x}$$

Here $\alpha(y_1, y_2) = \frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x} = x - 1$ which is non-zero. So Variation of Parameters can be used.

$$\begin{aligned} y_p &= \left| \begin{array}{c} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{array} \right| \\ &= \frac{e^{-2x}(1-x) + xe^{-x} \cdot e^{-x}}{x-1} \\ &= \frac{e^{-2x}[1-x+x]}{x-1} \\ &= \frac{e^{-2x}[1]}{x-1} \end{aligned}$$

$$= e^{-2x}[1]$$

$$\begin{aligned} &\text{Now } \frac{dy}{dx} \neq 0 \text{ so } \frac{dy}{dx} \neq 0 \\ &\text{So } \frac{dy}{dx} = 0 \text{ is not a solution of } \frac{dy}{dx} + 2 \frac{dy}{dx} + y = 0 \text{.} \end{aligned}$$

therefore y_1 and y_2 are linearly independent. \therefore let y_p be a particular solution of ①.

$$\frac{dy_p}{dx} = ue^{-x} + ve^{-x} \quad [\text{where } u \text{ and } v \text{ are the function of } x \text{ or constants}]$$

let us choose u and v in such a manner that

$$e^{-x} \frac{du}{dx} + \frac{dy}{dx} ve^{-x} = 0 \quad \text{--- ②}$$

therefore

$$\frac{dy_p}{dx} = -e^{-x}u + (1-x)e^{-x}v$$

$$\frac{d^2y_p}{dx^2} = -e^{-x} \frac{du}{dx} + (1-x)e^{-x} \frac{dv}{dx} + e^{-x}u + (x-1)e^{-x}v$$

Since y_p is a particular solution of ①

$$\text{Therefore } \frac{d^2y_p}{dx^2} + 2 \frac{dy_p}{dx} + y_p = \frac{e^x}{x^2} \quad [1. \text{ If } y_p = \frac{e^x}{x^2} \text{ then } \frac{dy_p}{dx} = \frac{e^x(x-1)}{x^3}, \frac{d^2y_p}{dx^2} = \frac{e^x(2x-3)}{x^4}]$$

$$-e^{-x} \frac{du}{dx} + (1-x)e^{-x} \frac{dv}{dx} + (1-x) \frac{e^{-x}}{x^2} = ① \quad \text{--- ③}$$

From ① and ③ we get \Rightarrow

$$-e^{-x} \frac{du}{dx} + (1-x)e^{-x} \frac{dv}{dx} + \frac{e^{-x}}{x^2} = 0 \quad [\text{since } v \text{ is a function of } x]$$

$$\frac{du}{dx} = \frac{-e^{-x} \frac{dv}{dx} + \frac{e^{-x}}{x^2}}{-e^{-x}} \quad = \frac{\frac{dv}{dx} - \frac{e^{-x}}{x^2}}{e^{-x}} \quad = \frac{\frac{dv}{dx} - e^{-x}(1/x)}{e^{-x}} \quad [1. \text{ If } v = \frac{e^x}{x^2} \text{ then } \frac{dv}{dx} = \frac{e^x(x-1)}{x^3}]$$

$$\frac{du}{dx} = -\frac{1}{x^2} \quad \text{--- ④}$$

$$\text{Integrating ④ we get } u = -\log x. \quad [\text{Let } x = e^t, \text{ then } \frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = u' \cdot 1]$$

$$\text{Integrating ④ we get } v = -\frac{1}{x} \quad \text{--- ⑤}$$

$$\text{Therefore } y_p = -\log x e^{-x} - \frac{1}{x} x e^{-x}$$

Then the required general solution is $y = y_p + \gamma_1 e^{x-2} + \gamma_2 e^{-x}$

$$= (9+ex) e^{-x} - e^{-x} (\log x + 1) + \gamma_1 e^{x-2} + \gamma_2 e^{-x}$$

$$\text{② solve by the method of variation of parameters } \frac{dy}{dx} + \alpha^2 y = \text{termax} \quad \text{--- ⑥}$$

$$\text{Let us first solve } \frac{dy}{dx} + \alpha^2 y = 0 \quad \text{--- ⑦}$$

$$\text{Let } y = e^{mx} \text{ be a trial solution of ⑦. Then the A.F. of ⑦ is } \rightarrow$$

$$m^2 + \alpha^2 = 0$$

$$\text{then } y_c = C_1 e^{mx} + C_2 \sin mx \quad [C_1, C_2 \text{ are arbitrary constants}]$$

Let $y_1 = \cos ax$ and $y_2 = \sin ax$
 Here $\omega(y_1, y_2) = \text{wronskian of } y_1, y_2$

$$= \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a \neq 0$$

Therefore y_1 and y_2 are linearly independent.

Let y_p be a particular solution of ①.

$y_p = u \cos ax + v \sin ax$ [Here u and v are function of x]

$$\frac{dy_p}{dx} = \frac{du}{dx} \cos ax + u \frac{d}{dx} \sin ax - v \sin ax \cdot a + av \cos ax$$

Let u & v choose u and v in such a manner that \Rightarrow ①

$$\frac{du}{dx} \cos ax + \sin ax \frac{dv}{dx} = 0 \quad \text{--- ②}$$

Therefore $\frac{dy_p}{dx} = au \sin ax + av \cos ax$

$$\frac{d^2y_p}{dx^2} = -a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - a^2 u \sin ax - a^2 v \cos ax$$

Since y_p is a particular solution of ①.
 Therefore,

$$\frac{d^2y_p}{dx^2} + a^2 y_p = \tan ax$$

$$-a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - a^2 u \sin ax - a^2 v \cos ax + a^2 u \sin ax + a^2 v \cos ax = \tan ax$$

$$-a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} = \tan ax \quad \text{--- ③}$$

From ② and ③ we get \rightarrow

$$-a \sin ax \frac{du}{dx} + a \cos ax \frac{dv}{dx} - \tan ax = 0$$

$$a \cos ax \frac{du}{dx} + \sin ax \frac{dv}{dx} + 0 = 0$$

$$\frac{du}{dx} = -\frac{1}{a} \frac{\sin ax}{\cos ax} \quad \text{--- ④} \quad \frac{dv}{dx} = -\frac{1}{a} \frac{\sin ax}{\cos ax} \quad \text{--- ⑤}$$

$$\int \frac{du}{dx} dx = -\frac{1}{a} \int \frac{\sin ax}{\cos ax} dx \quad \text{--- ④} \quad \int \frac{dv}{dx} dx = \frac{1}{a} \int \frac{\sin ax}{\cos ax} dx \quad \text{--- ⑤}$$

Integrating ④ we get $u = -\frac{1}{a^2} \log |\sec ax + \tan ax| + \frac{1}{a^2} \sin ax$

Integrating ⑤ we get $v = -\frac{1}{a^2} \cos ax$.

$$\text{Therefore } y_p = -\frac{1}{a^2} \log |\sec ax + \tan ax| \cos ax + \frac{1}{a^2} \sin ax \cos ax - \frac{1}{a^2} \cos ax \sin ax$$

$$= -\frac{1}{a^2} \cos ax \log |\sec ax + \tan ax|$$

Then the complete general solution is $y = y_c + y_p$

$$= q_1 \cos \omega x + q_2 \sin \omega x - \frac{d}{dt} \cos \omega x \log |\sec \omega x + \tan \omega x|$$

(12)

Galileo Galilei [1564-1642] – Italian astronomer
↳ A student of Galileo Galilei
↳ Author of "Two New Sciences" [1638] known as "Discorsi e Dimonstrazioni Matematiche intorno a due Novissime Scienze, la Mecanica & i Moti Locali et Celesti del Sistemo Pianetario"

References:-

① Differential Equation –
Haily & Mukherjee

② Differential Equation –
Haily & Mukherjee

↳ 2 methods of solution to eqn of 2nd
order

$$\text{Method 1: } \frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial b} \cdot \frac{\partial b}{\partial x}$$

$$\frac{\partial y}{\partial x} = \lambda \text{ constant} \Rightarrow \int \frac{\partial y}{\partial x} dx = \int \lambda dx$$

$$\therefore \frac{\partial y}{\partial x} = \lambda \Rightarrow y = \lambda x + C$$

$$y = \lambda x + C \quad \text{or} \quad \frac{dy}{dx} = \lambda x + C$$

$$\frac{dy}{dx} = \lambda x + C \quad \text{or} \quad \frac{dy}{dx} = \lambda x + C$$

$$y = \lambda x + C$$

$$\frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial b} \cdot \frac{\partial b}{\partial x}$$

$$\frac{\partial y}{\partial x} = \lambda \text{ constant} \Rightarrow \int \frac{\partial y}{\partial x} dx = \int \lambda dx$$

$$\therefore \frac{\partial y}{\partial x} = \lambda \Rightarrow y = \lambda x + C$$

$$y = \lambda x + C$$