- Learning Materials for Sem-2.

nit-1 , CC- 4

opic- Differential Equation

(Variation of Parameteres)

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① Solve by the method of variation of parameters: $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

$$\frac{d^2y}{dx^2} + a^2y = \sec ax \quad \text{—①}$$

Let us first solve $\frac{d^2y}{dx^2} + a^2y = 0$ —②

①

Let $y = e^{mx}$ be a trial solution of ②. Then the A.E of ② is →

$$e^{mx}(m^2+a^2) = 0$$

$$(m^2+a^2) = 0 \quad [\text{Since } e^{mx} \neq 0]$$

$$m = \pm ai$$

Then y_c = C.F of ①

$= c_1 \cos ax + c_2 \sin ax$ [c_1, c_2 are arbitrary constant]

Let $y_1 = \cos ax$, $y_2 = \sin ax$.

Here $w(y_1, y_2)$ = wronskain of y_1, y_2

$$= \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$$

$$= \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a \neq 0$$

Therefore y_1 and y_2 are linearly independent.

Let y_p be a particular solution of ①.

Let $y_p = u(x)\cos ax + v(x)\sin ax$.

$$\frac{dy_p}{dx} = \frac{du}{dx}\cos ax + \frac{dv}{dx}\sin ax - ua\sin ax + va\cos ax$$

Let us choose u and v in such a manner that

$$\frac{du}{dx}\cos ax + \frac{dv}{dx}\sin ax = 0 \quad \text{— (I)}$$

Therefore

$$\frac{dy_p}{dx} = va\cos ax - ua\sin ax.$$

$$\frac{d^2y_p}{dx} = -a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - a^2u\cos ax - a^2v\sin ax$$

Since y_p is a particular solution of ①, therefore →

$$\frac{d^2y_p}{dx^2} + a^2y_p = \sec ax$$

or, $-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - a^2u\cos ax - a^2v\sin ax + a^2u\cos ax + a^2v\sin ax = \sec ax$

or, $a\cos ax\frac{dv}{dx} - a\sin\frac{du}{dx} = \sec ax$

or, $-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - \sec ax = 0$ — (II)

From (I) and (II) we get →

$$-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - \sec ax = 0$$

$$\cos ax\frac{du}{dx} + \sin ax\frac{dv}{dx} - 0 = 0$$

Therefore $\frac{dy_p}{dx} = -e^{-x}u + (1-x)e^{-x}v$

(3)

$$\frac{d^2y_p}{dx^2} = e^{-x}u + (x-2)e^{-x}v - e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx}$$

Since y_p is a particular solution of ①.

Therefore $\frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + y_p = \frac{e^{-x}}{x}$

$$e^{-x}u + (x-2)e^{-x}v - e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} - 2e^{-x}u + 2(1-x)e^{-x}v + ue^{-x} + vxe^{-x} = \frac{e^{-x}}{x}$$

$$-e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} = \frac{e^{-x}}{x} \quad \text{— (II)}$$

From ① and (II) we get →

$$-e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} - \frac{e^{-x}}{x} = 0$$

$$e^{-x}\frac{du}{dx} + xe^{-x}\frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{e^{-2x}} = \frac{dv/dx}{-\frac{e^{-2x}}{x}} = \frac{1}{-e^{-2x}.x - e^{-2x}(1-x)}$$

$$= \frac{1}{-e^{-2x}[x+1-x]}$$

$$= \frac{1}{-e^{-2x}}$$

$$\frac{du}{dx} = -1 \quad \text{— ③} \qquad \frac{dv}{dx} = \frac{1}{x} \quad \text{— ④}$$

Integrating ③ we get $u = -x + c_3$

Integrating ④ we get $v = \log|x| + c_4$ [c_3, c_4 are arbitrary constant]

Therefore $y_p = (-x+c_3)e^{-x} + (\log|x| + c_4)xe^{-x}$

Then the required solution is →

$$y = y_p + y_c$$

$$= (c_1 + c_2x)e^{-x} + (-x+c_3)e^{-x} + (\log|x| + c_4)xe^{-x}$$

$$= e^{-x}c_1 + c_2xe^{-x} + c_3e^{-x} - xe^{-x} + \log|x|\,xe^{-x} + c_4xe^{-x}$$

$$= (c_1+c_3)e^{-x} + (c_2+c_4)xe^{-x} - xe^{-x} + \log|x|\,xe^{-x}$$

$$= c_5e^{-x} + c_6xe^{-x} - xe^{-x} + \log|x|\,xe^{-x}.$$

③ Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x, \quad x>0,$$ it is being given that $y = x$ and $y = \frac{1}{x}$ are two linearly independent solution of it reduced equation.

Since $y = x$ and $y = \frac{1}{x}$ are two linearly independent solution of ①. Therefore

$y_c = c_1x + c_2\frac{1}{x}$ [where c_1, c_2 are arbitary constant]

Here $y_1 = x$ and $y_2 = \frac{1}{x}$ are linearly independent solution.

Let y_p be a particular solution of ①.

Then $y_p = ux + \frac{v}{x}$ [where u and v are function of x or constant]

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$$\frac{dy_p}{dx} = \frac{du}{dx}\cdot x + u + \frac{dv}{dx}\frac{1}{x} - \frac{1}{x^2}v$$

Let us choose u and v in such a manner that

$$\frac{du}{dx}\cdot x + \frac{dv}{dx}\cdot\frac{1}{x} = 0 \quad \text{—— (I)}$$

Therefore. $\frac{dy_p}{dx} = u - \frac{1}{x^2}v$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx} - \frac{1}{x^2}\frac{dv}{dx} + \frac{2}{x^3}v$$

Since y_p is a particular solution of (I). Therefore

$$\frac{d^2y_p}{dx^2} + \frac{1}{x}\frac{dy_p}{dx} - \frac{1}{x^2}y_p = \log x.$$

$$\frac{du}{dx} - \frac{1}{x^2}\frac{dv}{dx} + \frac{2}{x^3}v + \frac{1}{x}u - \frac{1}{x^3}v - \frac{u}{x} - \frac{v}{x^3} = \log x$$

$$\frac{du}{dx} - \frac{1}{x^2}\frac{dv}{dx} = \log x \quad \text{—— (II)}$$

From (I) and (II) we get →

$$\frac{du}{dx} - \frac{1}{x^2}\frac{dv}{dx} - \log x = 0$$

$$x\frac{du}{dx} + \frac{1}{x}\frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{\frac{\log x}{x}} = \frac{dv/dx}{-x\log x} = \frac{1}{\frac{1}{x}+\frac{1}{x}}$$

$$= \frac{x}{2}$$

$$\frac{du}{dx} = \frac{\log x}{2} \quad \text{—— (3)} \qquad \frac{dv}{dx} = -\frac{1}{2}x^2\log x \quad \text{—— (4)}$$

Integrating (3) we get $u = \frac{1}{2}x + c_3$

Integrating (4) we get $v = -\frac{1}{6}(\log x - 1/3)x^3 + c_4$

Therefore $y_p = \left(\frac{1}{2}x + c_3\right)x + \left[-\frac{1}{6}(\log x - 1/3)x^3 + c_4\right]\frac{1}{x}$

$$= \frac{1}{2}x^2 - \frac{1}{6}(\log x - 1/3)x^2 + c_3x + c_4 \times \frac{1}{x}$$

Then the required general solution is →

$$y = y_p + y_c$$

$$= c_1x + c_2\frac{1}{x} + \frac{1}{2}x^2 - \frac{1}{6}(\log x - 1/3)x^2 + c_3x + \frac{c_4}{x}$$

$$= \frac{1}{2}x^2 - \frac{1}{6}(\log x - 1/3)x^2 + (c_1+c_3)x + (c_2+c_4)\frac{1}{x}$$

$$= \frac{1}{2}x^2 - \frac{1}{6}(\log x - 1/3)x^2 + c_4x + c_5\frac{1}{x}$$

(4) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x. \quad \boxed{2011}\text{ S.}$$

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \quad \text{—①}$$

5

Let us first solve $\frac{d^2y}{dx^2} + y = 0$ —②

Let $y = e^{mx}$ be a trial solution of ②. Then A.E of ② is →

$$e^{mx}(m^2+1) = 0$$

$$(m^2+1) = 0 \quad [\text{Since } e^{mx} \neq 0]$$

$$m = \pm i$$

Therefore y_c = C.F of ①

$$= c_1 \cos x + c_2 \sin x \quad [c_1, c_2 \text{ are arbitary constant}]$$

Let $y_1 = \cos x$ and $y_2 = \sin x$

Here $w(y_1, y_2)$ = wronskain of y_1, y_2

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= 1 \neq 0$$

Therefore y_1 and y_2 are linearly independent.

Let y_p be a particular solution of ①.

$y_p = u\sin x + v\cos x$ [Here u and v are function of x]

$$\frac{dy_p}{dx} = \sin x\frac{du}{dx} + \cos x \frac{dv}{dx} + u\cos x - v\sin x$$

Let us choose u and v in such a manner that

$$\sin x \frac{du}{dx} + \cos x \frac{dv}{dx} = 0 \quad \text{—(I)}$$

Then $\frac{dy_p}{dx} = u\cos x - v\sin x$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx}\cos x - \frac{dv}{dx}\sin x - u\sin x - v\cos x$$

$$= \frac{du}{dx}\cos x - \frac{dv}{dx}\sin x - (u\sin x + v\cos x)$$

Since y_p is a particular solution of ①. Therefore

$$\frac{d^2y_p}{dx^2} + y_p = \operatorname{cosec} x$$

$$\cos x \frac{du}{dx} - \frac{dv}{dx}\sin x - (u\sin x + v\cos x) + (u\sin x + v\cos x) = \operatorname{cosec} x$$

or, $\cos x \frac{du}{dx} - \sin x \frac{dv}{dx} = \operatorname{cosec} x$ —(II)

From (I) and (II) we get →

$$\sin x \frac{du}{dx} + \cos x \frac{dv}{dx} = 0$$

$$\cos x \frac{du}{dx} - \sin x \frac{dv}{dx} = \operatorname{cosec} x$$

$$\frac{\frac{du}{dx}}{-\cot x} = \frac{dv/dx}{1} = \frac{1}{-1}$$

$\frac{du}{dx} = \cot x$ —— ③ $\quad \frac{dv}{dx} = -1$ —— ④

Integrating ③ we get $u = \log(\sin x) + c_3$

Integrating ④ we get $v = -x + c_4$

Therefore $y_p = (\log(\sin x) + c_3)\sin x + (-x + c_4)\cos x$

Then the required general solution is $y = y_p + y_c$

$$= c_1\cos x + c_2\sin x + [\log(\sin x) + c_3]\sin x + (-x + c_4)\cos x$$

$$= c_1\cos x + c_4\cos x + c_2\sin x + c_3\sin x + [\log(\sin x)]\sin x - x\cos x$$

$$= \cos x(c_1 + c_4) + \sin x(c_2 + c_3) + \log(\sin x)\sin x - x\cos x$$

$$= c_5\cos x + c_6\sin x + [\log(\sin x)]\sin x - x\cos x.$$

5. Solve by the method of variation of parameters $x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y$ given that $y = x$ and $y = xe^x$ are independent solution of the reduced equation.

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = x^3$$

$$\frac{d^2y}{dx^2} - \left(\frac{x+2}{x}\right)\frac{dy}{dx} + \left(\frac{x+2}{x^2}\right)y = x \quad \text{— ①}$$

Since $y = x$ and $y = xe^x$ are two linearly independent solution of ①.

Therefore → $y_c = c_1x + c_2xe^x$ [c_1 and c_2 are arbitrary constant]

Let y_p be a particular solution of ①

Then $y_p = ux + vxe^x$ [where u and v are function of x or constant]

$$\frac{dy_p}{dx} = \frac{du}{dx}\cdot x + \frac{dv}{dx}xe^x + u + v(x+1)e^x$$

Let us choose u and v in such a manner that →

$$\frac{du}{dx}\cdot x + \frac{dv}{dx}xe^x = 0 \quad \text{— I}$$

Therefore $\frac{dy_p}{dx} = u + v(x+1)e^x$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx} + \frac{dv}{dx}(x+1)e^x + ve^x(x+2)$$

Since y_p is a particular solution of ①. Therefore

$$\frac{d^2y_p}{dx^2} - \left(\frac{x+2}{x}\right)\frac{dy_p}{dx} + \frac{x+2}{x}y_p = x$$

$$\frac{du}{dx} + \frac{dv}{dx}(x+1)e^x = x \quad \text{— II}$$

From I and II we get →

$$\frac{\frac{du}{dx}}{x^2e^x} = \frac{dv/dx}{x^2} = \frac{1}{xe^x - xe^x(x+1)}$$

$$= \frac{1}{-x^2e^x}$$

$\frac{du}{dx} = -1$ —— ③ $\quad \frac{dv}{dx} = e^{-x}$ — ④

Integrating ③ we get $u = -x + c_3$

Integrating ④ we get $v = -e^{-x} + c_4$

Therefore

$$y_p = -x \cdot x - e^{-x} \cdot x e^{x} + c_3 + c_4$$
$$= -x^2 - x + c_5$$
$$= -(x^2 + x) + c_5$$

Therefore the required general solution is →

$$y = y_p + y_c$$
$$= -x^2 - x + c_1 x + c_2 x e^{x} + c_5$$
$$= -x^2 - x + c_6 x + c_7 x e^{x}$$

6. Solve by the method of variation of parameters $(1+x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1+x)^2$
It is given that $y = x$ and $y = e^{-x}$ are independent solution of the reduced equation.

$$(1+x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1+x)^2$$

$$\frac{d^2y}{dx^2} + \frac{x}{1+x}\frac{dy}{dx} - \frac{1}{1+x}y = (1+x) \text{ — ①}$$

Since $y = x$ and $y = e^{-x}$ are two linearly independent solution of ①.

Therefore $y_c = c_1 x + c_2 e^{-x}$ [c_1 and c_2 are A.C]

Let y_p be a particular solution of ①

Then $y_p = ux + ve^{-x}$ [where u and v are function of x or constant]

$$\frac{dy_p}{dx} = \frac{du}{dx}\cdot x + \frac{dv}{dx}e^{-x} + u - e^{-x}v$$

Let us choose u and v in such a manner that →

$$\frac{du}{dx}\cdot x + \frac{dv}{dx}\cdot e^{-x} = 0 \text{ —— ①}$$

Therefore $\frac{dy_p}{dx} = u - e^{-x}v$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx} - e^{-x}\frac{dv}{dx} + e^{-x}v$$

Since y_p be a particular solution of ①, therefore

$$\frac{d^2y_p}{dx^2} + \frac{x}{1+x}\frac{dy_p}{dx} - \frac{1}{1+x}y_p = 1+x$$

$$\frac{du}{dx} - e^{-x}\frac{dv}{dx} = 1+x \text{ —— (II)}$$

From ① and (II) we get →

$$\frac{du}{dx} - e^{-x}\frac{dv}{dx} - (1+x) = 0$$

$$x\frac{du}{dx} + e^{-x}\frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{e^{-x}(1+x)} = \frac{\frac{dv}{dx}}{-x(1+x)} = \frac{1}{e^{-x}+e^{-x}.x}$$

$$= \frac{1}{e^{-x}(1+x)}$$

(8)

$\frac{du}{dx} = 1$ —— ③ $\quad$ $\frac{dv}{dx} = \frac{-x}{e^{-x}}$ — ④

Integrating ③ we get $u = x$

Integrating ④ we get $v = e^{x}(1-x)$

Therefore $y_p = x^2 + (1-x)e^{-x}.e^{x}$

$= x^2 + 1 - x$

∴ Therefore the required general solution is $y = y_p + y_c$

$= c_1 x + c_2 e^{-x} + 1 - x + x^2$

7. Solve by the method of variation of parameters $(2x+1)(1+x)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 2y +$ It being given that $y = x$ and $y = \frac{1}{1+x}$ are two linearly independent solution of the corrosponding homogeneous equation.

$$(2x+1)(1+x)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = (2x+1)^2$$

$$\frac{d^2y}{dx^2} + \frac{2x}{(2x+1)(1+x)}\frac{dy}{dx} - \frac{2}{(2x+1)(1+x)}y = \frac{(2x+1)}{(x+1)} \quad \text{— ①}$$

Since $y = x$ and $y = \frac{1}{x+1}$ are two linearly independent solution of ①. Therefore →

$y_c = c_1 x + c_2 \frac{1}{x+1}$ [c_1 and c_2 are arbitary constant]

Let y_p be a particular solution of ①

Then $y_p = ux + \frac{v}{x+1}$ [where u and v are function of x or constant]

$$\frac{du}{dx}x + \frac{dv}{dx}\cdot\frac{1}{x+1} = 0 \quad \text{—— (I)}$$

Therefore $\frac{dy_p}{dx} = u + \frac{v}{-(x+1)^2}$

$$\frac{d^2y_p}{dx^2} = \frac{du}{dx} - \frac{dv}{dx}\left(\frac{1}{x+1}\right)^2 + \frac{1}{(x+1)^3}v$$

Since y_p is a particular solution. Therefore

$$\frac{d^2y_p}{dx^2} + \frac{2x}{(2x+1)(x+1)}\frac{dy_p}{dx} - \frac{2}{(2x+1)(x+1)}y_p = \frac{(2x+1)}{(x+1)}$$

$$\frac{du}{dx} - \frac{1}{(x+1)^2}\frac{dv}{dx} = \frac{2x+1}{(x+1)} \quad \text{—— (II)}$$

From (I) and (II) we get →

$$\frac{du}{dx} - \frac{1}{(x+1)^2}\frac{dv}{dx} - \frac{2x+1}{x+1} = 0$$

$$x\frac{du}{dx} + \frac{1}{(x+1)}\frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{\frac{2x+1}{(x+1)^2}} = \frac{\frac{dv}{dx}}{\frac{-x(2x+1)}{x+1}} = \frac{1}{\frac{1}{x+1}+\frac{x}{(x+1)^2}}$$

$$= \frac{(x+1)^2}{x+1+x}$$

$$= \frac{(x+1)^2}{2x+1}$$

$\frac{du}{dx} = 1$ —— ③ $\quad \frac{dv}{dx} = -(x^2+x)$ —— ④

Integrating ③ we get $u = x$

Integrating ④ we get $v = -\frac{1}{6}(2x^3+3x^2)$

Therefore $y_p = \frac{x^2}{6}\cdot\frac{4x+3}{x+1}$

Then the required general solution is $y = y_p + y_c$

$$= \frac{x^6}{6}\cdot\frac{4x+3}{x+1} + c_1x + \frac{c_2}{x+1}$$

⑧ Solve by variation of parameters : $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}$
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$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2} \quad \text{—①}$$

Let us first solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ —②

Let $y = e^{mx}$ be a trial solution of ②. Then the A.E of ② is →

$m^2 + 2m + 1 = 0$ [Since $e^{mx} \neq 0$]

$(m+1)^2 = 0$

$m = -1, -1$

Therefore $y_c = (c_1 + c_2x)e^{-x}$ [c_1, c_2 are arbitary constant]

Let $y_1 = e^{-x}$ and $y_2 = xe^{-x}$

Here $w(y_1, y_2)$ = wronskain of y_1, y_2

$$= \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$$

$$= e^{-2x}(1-x) + xe^{-x}\cdot e^{-x}$$

$$= e^{-2x}[1-\cancel{x}+\cancel{x}]$$

$$= e^{-2x}[1]$$

$$= e^{-2x} \neq 0$$

Therefore y_1 and y_2 are linearly independent.

Let y_p be a particular solution of ①.

Then $y_p = ue^{-x} + vxe^{-x}$ [where u and v are the function of x or constant]

$$\frac{dy_p}{dx} = e^{-x}\frac{du}{dx} + \frac{dv}{dx}ve^{-x} - e^{-x}u + (1-x)e^{-x}v$$

Let us choose u and v in such a manner that

$$e^{-x}\frac{du}{dx} + \frac{dv}{dx}xe^{-x} = 0 \quad \text{—— (I)}$$

Therefore

$$\frac{dy_p}{dx} = -e^{-x}u + (1-x)e^{-x}v$$

$$\frac{d^2y_p}{dx^2} = -e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} + e^{-x}u + (x-2)e^{-x}v$$

Since y_p is a particular solution of ①

Therefore $\frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + y_p = \frac{e^{x}}{x^2}$

$$-e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} = \frac{e^{-x}}{x^2} \quad \text{—— (II)}$$

From (I) and (II) we get →

$$-e^{-x}\frac{du}{dx} + (1-x)e^{-x}\frac{dv}{dx} - \frac{e^{-x}}{x^2} = 0$$

$$e^{-x}\frac{du}{dx} + xe^{-x}\frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{\frac{e^{-2x}}{x}} = \frac{dv/dx}{\frac{-e^{-2x}}{x^2}} = \frac{1}{-e^{-2x}.x - e^{-2x}(1-x)}$$

$$= \frac{1}{-e^{-2x}}$$

$$\frac{du}{dx} = -\frac{1}{x} \quad \text{—— ③} \qquad \frac{dv}{dx} = \frac{1}{x^2} \quad \text{—— ④}$$

Integrating ③ we get $u = -\log x$.

Integrating ④ we get $v = -\frac{1}{x}$

Therefore $y_p = -\log x . e^{-x} - \frac{1}{x}xe^{-x}$

Then the required general solution is $y = y_p + y_c$

$$= (c_1 + c_2x)e^{-x} - e^{-x}(\log x + 1)$$

⑨ Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + a^2y = \tan ax$. [2010]

$$\frac{d^2y}{dx^2} + a^2y = \tan ax \quad \text{—— ①}$$

Let us first solve $\frac{d^2y}{dx^2} + a^2y = 0$ —— ②

Let $y = e^{mx}$ be a trial solution of ②. Then the A.E of ② is →

$$m^2 + a^2 = 0$$

$$m = \pm ai$$

Then $y_c = c_1\cos ax + c_2\sin ax$ [c_1, c_2 are arbitrary constant]

(11)

Let $y_1 = \cos ax$ and $y_2 = \sin ax$

Here $w(y_1, y_2)$ = wronskian of y_1, y_2

$$= \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$= a \neq 0$

Therefore y_1 and y_2 are linearly independent.

Let y_p be a particular solution of (1).

$y_p = u\cos ax + v\sin ax$ [Here u and v are function of x]

$$\frac{dy_p}{dx} = \frac{du}{dx}\cos ax + \frac{dv}{dx}\sin ax - u\sin ax \cdot a + av\cos ax$$

Let us choose u and v in such a manner that →

$$\frac{du}{dx}\cos ax + \sin ax\frac{dv}{dx} = 0 \quad \text{— (I)}$$

Therefore $\frac{dy_p}{dx} = -au\sin ax + av\cos ax$

$$\frac{d^2y_p}{dx^2} = -a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - a^2u\cos ax - a^2v\sin ax$$

Since y_p is a particular solution of (1).

Therefore,

$$\frac{d^2y_p}{dx^2} + a^2y_p = \tan ax$$

$$-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - a^2u\cos ax - a^2v\sin ax + a^2u\cos ax + a^2v\sin ax = \tan ax$$

$$-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} = \tan ax \quad \text{— (II)}$$

From (I) and (II) we get →

$$-a\sin ax\frac{du}{dx} + a\cos ax\frac{dv}{dx} - \tan ax = 0$$

$$\cos ax\frac{du}{dx} + \sin ax\frac{dv}{dx} + 0 = 0$$

$$\frac{\frac{du}{dx}}{\frac{\sin^2 ax}{\cos ax}} = \frac{dv/dx}{-\sin ax} = \frac{1}{-a}$$

$$\frac{du}{dx} = -\frac{1}{a}\frac{\sin^2 ax}{\cos ax} \quad \text{— (3)} \qquad \frac{dv}{dx} = \frac{1}{a}\sin ax \quad \text{— (4)}$$

Integrating (3) we get $u = -\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax$

Integrating (4) we get $v = -\frac{1}{a^2}\cos ax$.

Therefore $y_p = -\frac{1}{a^2}\log|\sec ax + \tan ax|\cos ax + \frac{1}{a^2}\cos ax\sin ax - \frac{1}{a^2}\cos ax\sin ax$

$= -\frac{1}{a^2}\cos ax\log|\sec ax + \tan ax|$

Then the complete general solution is $y = y_c + y_p$

$= c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log|\sec ax + \tan ax|$

References:–

1. Differential Equation – Bej & Mukherjee.

2. Differential Equation – Maity & Ghosh.