2nd week.

Notes on Differential Equ$^n$.
for Sem 2 Math Hons Paper C-4T
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Super Position Principle: Fundamental Theorem of homogeneous ODE (ordinary diff. Equ$^n$) $(D^2+PD+Q)y=0$ or $f(D)y=0$
ie $y''+Py'+Qy=0$ ... ① If $y_1(x)$, $y_2(x)$ be the two L.I. sol$^n$s of ① then the linear combination of these two solution again be the solution of ①.
(linearly independent L.I.)
These principle holds good only for homogeneous diff. Equ$^n$ but not for non homogeneous equ$^n$.

* If $y_1(x)$; $y_2(x)$ ... $y_n(x)$ be the $n$ solutions linearly independent of the diff. Equ$^n$ $\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy}{dx}+P_ny=0$ ... ②
or $f(D)y=0$
then $y=c_1y_1+c_2y_2+\cdots+c_ny_n$ [is linear combination of the $n$ solution] is also an integral or Solution of ②. where $y_1, y_2 \cdots y_n(x)$ be the independent solutions. As order of ② is $n$ so there will be $n$ arbitrary Consts.
This concept often called Super Position Principle or linearity Principle only for linearly independent sol$^n$s.
For 2nd order homogeneous diff. equ$^n$. $D^2y+PDy+Qy=0$
it will be $y=c_1y_1+c_2y_2$ two arbitrary Constants $c_1, c_2$
and ~~$y_1(x), y_2(x)$~~ $y_1(x)$, $y_2(x)$ be the two independent sol$^n$s.

For non homogeneous equation ~~$D^n+P_1$~~
$(D^n+P_1D^{n-1}+\cdots+P_{n-1}D+P_n)y=R(x)$
its solution has two parts: Homogeneous Diff. Equ$^n$ will be obtained by putting $R(x)=0$ ~~Complimentary function~~
By trial sol$^n$. $y=e^{mx}$ Auxiliary Equation will be obtained. values of $m$ will be found. By Rule C.F will be obtained is the 1st Part of the solution

2nd Part of the solution is Particular Integral (P.I.) without any ~~ab~~ arbitrary constant.

$P.I = \frac{1}{f(D)} R(x)$ ~~where $f(D) = D^n + P_1 D^{n-1} + \dots + P_n D$~~

where $f(D) = D^n + P_1 D^{n-1} + \dots + P_{n-1} D + P_n$ ~~= 0~~.

for 2nd order $f(D) = D^2 + P_1 D + q$ [ D is differential operator ]

Complete solution = complementary function + particular integral,
$= C.F + P.I.$

Existence of linearly independence is verified by ~~wor~~ wronskian. If $y_1(x), y_2(x)$ be any sol$^n$s.

~~where~~ then $W(y_1, y_2, x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ maybe $W(y_1, y_2, x)$
(wronskian determinant)

if $W(y_1, y_2) \neq 0$ then $y_1, y_2$ sol$^n$s are linearly independent.
otherwise not linearly independent
or (Linearly Dependent).

EX. Verify the solutions of D.E $y'' - 5y' + 6y = 0$ are linearly independent?
$\left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0\right)$ $\left[y'' = \frac{d^2y}{dx^2}, \; y' = \frac{dy}{dx}\right]$

sol$^n$:- Its A.E is $m^2 - 5m + 6 = 0$.
$\Rightarrow (m-3)(m-2) = 0 \Rightarrow m = 3, 2$

Two sol$^n$s are $y_1 = e^{2x}$, $y_2 = e^{3x}$

So $W(y_1, y_2, x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x}$

so $e^{5x} \neq 0$.

hence these two sol$^n$ are L. I. (linearly independent)

Solution of ~~D.E. of~~ linear homogeneous diff. equ$^n$ of non-homogeneous of higher order with constant coefficients.

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**Ex 1.** $(D^4 - 5D^2 + 4)y = 0$ This ~~equ~~ D.E. is of the form

$$\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$$ its A.E is

$m^4 - 5m^2 + 4 = 0$ by a trial sol$^n$ $y = e^{mx}$

or, $(m^2-4)(m^2-1) = 0 \Rightarrow m = 2, -2, 1, -1$

So C.F or general solution (as it is homogeneous D.E.)

is $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{-x}$, $c_1, c_2, c_3, c_4$ being arbitrary constants.

**Ex 2** solve $\frac{d^4y}{dx^4} + a^4 y = 0$.

Sol$^n$. The Auxiliary equ$^n$ is $m^4 + a^4 = 0$.

or $(m^2+a^2)^2 - (\sqrt{2}\,am)^2 = 0$

or, $(m^2+a^2+\sqrt{2}\,am)(m^2+a^2-\sqrt{2}\,am) = 0$

either $m^2+a^2+\sqrt{2}\,am = 0$ or, $m^2+a^2-\sqrt{2}\,am = 0$

hence $m = \frac{-\sqrt{2}a \pm \sqrt{2a^2 - 4\cdot a^2}}{2}$ and $m = \frac{\sqrt{2}a \pm \sqrt{2a^2-4a^2}}{2}$

$= -\frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$ $\qquad = \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$

Hence the general sol$^n$ is

$$y = e^{-\frac{ax}{\sqrt{2}}}\left\{c_1\cos\left(\frac{ax}{\sqrt{2}}\right) + c_2\sin\left(\frac{ax}{\sqrt{2}}\right)\right\} + e^{\frac{ax}{\sqrt{2}}}\left\{c_3\cos\left(\frac{ax}{\sqrt{2}}\right) + c_4\sin\left(\frac{ax}{\sqrt{2}}\right)\right\}$$

$c_1, c_2, c_3, c_4$ being arbitrary constants.

Consult Diff. Equation 1) M.D. Raisinghania
2) Maity Ghosh.

To solve the non-homogeneous diff. equ$^n$. we have to know how to find the particular integral. because it has two parts 1st part is C.F (Complementary function) which is found early equating ~~makup~~ the R.H.S. of D.E. to Zero.

Now Particular integral $= \frac{1}{f(D)} R(x)$ or, $\frac{1}{f(D)} \cdot f(x)$ etc

Results on symbolic operator

(i) $f(D) e^{ax} = f(a) e^{ax}$ where $f(D)$ be any rational integral function of D.

(ii) $f(D) e^{ax} \cdot v = e^{ax} f(D+a) v$, where v is function of $x$

~~ect~~ etc.

again $D(x^2) = 2x$, $D(\sin x) = \cos x$.

$\frac{1}{D}$ is the inverse oparator s.t.

$\frac{1}{D} 2x = x^2$, $\frac{1}{D^2} x = \frac{1}{D} \cdot \frac{1}{D} x = \frac{1}{D}\left(\frac{1}{2}x^2\right) = \frac{1}{2} \cdot \frac{1}{3} x^3 = \frac{1}{6} x^3$

$\frac{1}{D^2}(\cos x) = -\cos x$. etc.

Rule: (1) $\frac{1}{f(D)} e^{ax} = e^{ax} \cdot \frac{1}{f(a)}$. if $f(a) \neq 0$ ~~then it fails~~ and say

so $\boxed{\cancel{\frac{1}{f(D)} e^{ax}}} =$ if $f(a) = 0$

(2) $f(D) = (D-a)^n$, $n = 1, 2, \ldots$

Then $P.I = \frac{1}{(D-a)^n} \cdot e^{ax} = e^{ax} \cdot \frac{1}{(D+a-a)^n} \cdot 1 = e^{ax} \frac{1}{D^n} \cdot 1$

again if $f(D) = (D-a)^r \cdot \phi(D)$ where $\phi(a) = 0$.

then $P.I = \frac{1}{(D-a)^r} \cdot \frac{1}{\phi(D)} \cdot e^{ax} = \frac{1}{\phi(a)} \cdot \frac{1}{(D-a)^r} \cdot e^{ax}$

$= \frac{1}{\phi(a)} \cdot e^{ax} \cdot \frac{x^r}{r!}$

To find the P.I of $\frac{1}{\phi(D^2)}\sin ax = \frac{\sin ax}{\phi(-a^2)}$

$\frac{1}{\phi(D^2)}\cos ax = \frac{\cos ax}{\phi(-a^2)}$ Put $D^2=-a^2$ if $\phi(-a^2)\neq 0$.

if $\phi(-a^2)=0$ then $\frac{1}{\phi(D^2)}\sin ax = x\cdot\frac{\sin ax}{\phi'(-a^2)}$

again if $\phi'(-a^2)=0$ then $\frac{1}{\phi(D^2)}\sin ax = x^2\,\frac{\sin ax}{\phi''(-a^2)}$

etc Consult Book.

To find P.I. of $\frac{1}{D^2+a^2}\cos ax$

$=$ Real part of $\frac{1}{D^2+a^2}e^{iax}$.

P.I. of $\frac{1}{D^2+a^2}\sin ax$ = Imaginary part of $\frac{1}{D^2+a^2}e^{iax}$

Rule. $\frac{1}{f(D)}e^{ax}\cdot v = e^{ax}\frac{1}{f(D+a)}\cdot v$ (where v is function of x)

Find P.I. of $\frac{1}{D^2+9}(x\sin x)$ = imaginary part of $\frac{1}{D^2+9}xe^{ix}$

$=$ I.P. of $e^{ix}\frac{1}{(D+i)^2+9}x$ by $\frac{1}{f(D)}\cdot v\,e^{ax} = \frac{1}{f(D+a)}v$

$=$ I.P. of $e^{ix}\cdot\frac{1}{D^2+2iD+8}x$ = I.P. of $\frac{e^{ix}\cdot 1}{8\left(1+\frac{D^2+2iD}{8}\right)}\cdot x$

$=$ I.P. of $\frac{e^{ix}}{8}\left[1+\frac{iD}{4}+\frac{D^2}{8}\right]^{-1}x$

$=$ I.P. of $\frac{e^{ix}}{8}\left(1-\frac{iD}{4}+\cdots\right]x$ as $D^2(x)=0$. neglecting $D^2$ & higher power

imaginary part (I.P.) $=$ I.P. of $\frac{1}{8}(\cos x+i\sin x)\left(x-i\frac{1}{4}\right)$

$=\frac{1}{8}\left\{x\sin x-\frac{1}{4}\cos x\right\}$

**Ex** Solve $(D^2+2D+1)y = xe^{x}\sin x$ or find general solutions.

Sol$^n$: For the C.F. Let us solve the equation

$(D^2+2D+1)y=0$. So Auxiliary equ$^n$ is

$m^2+2m+1=0$ if $y=e^{mx}$ be a trial sol$^n$.

$m = 1, 1$

$\therefore$ C.F. $= (c_1+c_2x)e^{x}$, $c_1$, $c_2$ being arbitrary constants

$$P.I. = \frac{1}{(D-1)^2} e^{x}(x\sin x) = e^{x} \cdot \frac{1}{(D+1-1)^2} x\sin x$$

$$= e^{x} \frac{1}{D^2}(x\sin x) = e^{x} \cdot \frac{1}{D} \cdot \frac{1}{D}(x\sin x)$$ integrating by parts

$$= e^{x} \frac{1}{D}\int x\sin x\,dx = e^{x}\frac{1}{D}(-x\cos x+\sin x)$$

$$= e^{x}\int(\sin x - x\cos x)dx = e^{x}\left[\int \sin x\,dx - \int x\cos x\,dx\right]$$

$$= e^{x}\left[-\cos x - \{x\sin x + \cos x\}\right] = e^{x}(-x\sin x - 2\cos x)$$

The required general sol$^n$ = $y_c + y_p$

$$\therefore\ y = (c_1+c_2x)e^{x} - e^{x}(x\sin x + 2\cos x)$$

**Ex** Find the complete solution of $(D^3+1)y = \cos^2 x/_2 + e^{-x}$

Sol$^n$: For C.F Let $(D^3+1)y=0$. its

A.E is $m^3+1=0$ by a trial sol$^n$ $y=e^{mx}$

$\therefore (m+1)(m^2-m+1)=0 \Rightarrow m=-1,\ m=\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

So C.F $= c_1e^{-x} + e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$.

Now for P.I.

$P.I = \frac{1}{D^3+1}\left[\cos^2\frac{x}{2} + e^{-x}\right] = \frac{1}{D^3+1}\cos^2\frac{x}{2} + \frac{1}{D^3+1}e^{-x}$

$= \frac{1}{D^3+1}\left(\frac{1+\cos x}{2}\right) + \cancel{\frac{1}{(D-1)^3+1}e^{-x}} + \frac{1}{(D+1)(D^2-D+1)}\cdot e^{-x}$

$= \frac{1}{2}\frac{1}{D^3+1}e^{0x} + \frac{1}{2}\frac{1}{D^2\cdot D+1}\cos x + \frac{1}{(-1)^2-(-1)+1}\cdot\frac{1}{D-1+1}\cdot e^{-x}$

$= \frac{1}{2} + \frac{1}{2}\frac{1}{-D+1}\cdot\cos x + \frac{1(-1)}{3}e^{-x}$

$= \frac{1}{2} + \frac{1}{2}\frac{(1+D)}{1-D^2}\cdot\cos x - \frac{1}{3}e^{-x} \quad [D^2=-1]$

$= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}(\cos x - \sin x) - \frac{1}{3}e^{-x}$

$= \frac{1}{2} + \frac{1}{4}\cos x - \frac{1}{4}\sin x - \frac{1}{3}e^{-x}$

$= \frac{1}{2} + \frac{1}{4}(\cos x - \sin x) - \frac{1}{3}e^{-x}$

Hence the complete soln is

$y = c_1 e^{-x} + e^{x/2}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{2} + \frac{1}{4}(\cos x - \sin x) - \frac{1}{3}e^{-x}$

<u>EX</u> Find the complete solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$$

Soln:- To find C.F. let $(D^2-3D+2)y=0$ by trial soln $y=e^{mx}$

its A.E is $m^2-3m+2=0$,

$\Rightarrow m = 1, 2 \quad \therefore CF = c_1e^{x} + c_2e^{2x}$

$P.I = \frac{1}{D^2-3D+2}(xe^{3x} + \sin 2x) = \frac{1}{D^2-3D+2}x\cdot e^{3x} + \frac{1}{D^2-3D+2}\sin 2x$

$= e^{3x}\cdot\frac{1}{(D+3)^2-3(D+3)+2}\cdot x + \frac{1}{-4-3D+2}\sin 2x$

$= e^{3x}\cdot\frac{1}{D^2+6D+9-3D-9+2}\cdot x + \frac{1}{-3D-2}\sin 2x$

$$= \frac{e^{3x}}{2}\left[1+\frac{3D+D^2}{2}\right]^{-1} x - \frac{3D-2}{9D^2-4}\sin 2x$$

Put $\left[D^2=-4\right.$

$$= \frac{e^{3x}}{2}\left[1-\left(\frac{3D+D^2}{2}\right)+\cdots\right] x - \frac{(3D-2)}{9(-4)-4}\sin 2x$$

$$= \frac{1}{2}e^{\frac{3x}{2}}\left[x - \frac{3D+D^2}{2}x + \cdots\right] + \frac{3}{40}D(\sin 2x) - \frac{\sin 2x}{20}$$

$$= \frac{1}{2}e^{\frac{3x}{2}}\left(\frac{2x-3}{2}\right) + \frac{3}{20}\cos 2x - \frac{1}{20}\sin 2x$$

Hence complete solution is

$$y = CF + P.I.$$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{4}(2x-3) + \frac{3}{20}\cos 2x - \frac{1}{20}\sin 2x.$$

**Ex.** Prove that each of the functions $x, x^2, x^4$ are independent solutions of $x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 8y = 0$, write the general solution.

Sol$^n$: Now putting $y = x, x^2$, and $x^4$ we get

(1) $x^3\frac{d^3}{dx^3}(x) - 4x^2\frac{d^2}{dx^2}(x) + 8x\frac{d}{dx}(x) - 8\cdot x$

$$= x^3\times 0 - 4x^2\times 0 + 8x - 8x = 0$$

(2) $x^3\frac{d^3}{dx^3}(x^2) - 4x^2\frac{d^2}{dx^2}(x^2) + 8x\frac{d}{dx}(x^2) - 8x^2$

$$= x^3\times 0 - 4x^2\times 2 + 8x(2x) - 8x^2 = 0$$

(3) $x^3\left(\frac{d^3}{dx^3}\right)(x^4) - 4x^2\frac{d^2}{dx^2}(x^4) + 8x\frac{d}{dx}(x^4) - 8x^4 = 0.$

hence, $x, x^2, x^4$ are solutions. Their Wronskian $W(x, x^2, x^4) = \begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix}$

$$= x\begin{vmatrix} 2x & 4x^3 \\ 2 & 12x^2 \end{vmatrix} - x^2\begin{vmatrix} 1 & 4x^3 \\ 0 & 12x^2 \end{vmatrix} + x^4\begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix}$$ simplify it

$= 6x^4 \neq 0$ so $x, x^2, x^4$ are linearly independent solutions of the given differential equation.

The general solution is $y = c_1x + c_2x^2 + c_3x^4$, $c_1, c_2, c_3$ are arbitrary constants.

Ex. Given $y=x$ is a solution of the D.E.
$$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0 \quad (1), \quad 0<x<\infty,$$ then find another independent solution.

Sol$^n$:- Let the transformation $y=vx$ (Let $v$ is function of $x$)

Then $\frac{dy}{dx}=x\frac{dv}{dx}+v$ and $\frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+\frac{dv}{dx}+\frac{dv}{dx}$
$$=x\frac{d^2v}{dx^2}+2\frac{dv}{dx}$$

Now the given diff. equ$^n$ (1) is transformed to
$$(x^2+1)\left(x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right)-2x\left(x\frac{dv}{dx}+v\right)+2vx=0$$
$$\text{or } x(x^2+1)\frac{d^2v}{dx^2}+2x^2\frac{dv}{dx}+2\frac{dv}{dx}-2x^2\frac{dv}{dx}-2vx+2vx=0$$
$$\text{or, } x(x^2+1)\frac{d^2v}{dx^2}+2\frac{dv}{dx}=0 \quad \text{--- (2)}$$

Put $\frac{dv}{dx}=w \Rightarrow \frac{dw}{dx}=\frac{d^2v}{dx^2}$

Hence (2) becomes $x(x^2+1)\frac{dw}{dx}+2w=0$ --- (3)

which is 1st order differential equ$^n$.

from (3) we have
$$\frac{dw}{w}=-\frac{2}{x(x^2+1)}dx \quad \text{integrating}$$
$$\log w=\int\frac{-2\,dx}{x(x^2+1)}=-2\int\frac{x^2+1-x^2}{x(x^2+1)}dx$$
$$\log w=-2\left\{\int\frac{dx}{x}-\int\frac{x}{x^2+1}dx\right\}$$
$$\therefore \log w=-2\left\{\log x-\frac{1}{2}\log(x^2+1)\right\}+\log c$$
$$\text{or } \log w=\log\frac{c(x^2+1)}{x^2} \Rightarrow w=\frac{c(x^2+1)}{x^2}, \quad \text{Put } c=1$$

one Particular solution is $w=\frac{x^2+1}{x^2}$ or, $\frac{dv}{dx}=\frac{x^2+1}{x^2}$ $[w=$

or, $\int dv=\int\frac{x^2+1}{x^2}dx \Rightarrow v(x)=x-\frac{1}{x}$

another sol$^n$ is $y=v(x)\times x \Rightarrow y=x\left(x-\frac{1}{x}\right)=x^2-1$

Now wronskian $W(x, x^2-1)=\begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix}=x^2+1\neq 0.$

Hence the two solutions $y=x$ and $y=x^2-1$ are independent

the general solution is $y=c_1x+c_2(x^2-1)$.

Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0$ (1) given that $y=e^x$ is an integral included in the C.F.

Sol$^n$:— ~~Let~~ Putting $y=ve^x$ in given eq$^n$. (1)

$\therefore \frac{dy}{dx}=ve^x+e^x\frac{dv}{dx}$

$\therefore \frac{d^2y}{dx^2}=ve^x+e^x\frac{dv}{dx}+e^x\frac{d^2v}{dx^2}+e^x\frac{dv}{dx}$

(1) reduces to $x\left(ve^x+2e^x\frac{dv}{dx}+e^x\frac{d^2v}{dx^2}\right)$

$-(2x-1)\left(ve^x+e^x\frac{dv}{dx}\right)+(x-1)ve^x=0$

dividing by $e^x$ we get $e^x\neq 0$.

$\cancel{xv}+\cancel{2x\frac{dv}{dx}}+x\frac{d^2v}{dx^2}-\cancel{2xv}-\cancel{2x\frac{dv}{dx}}+v+\frac{dv}{dx}+\cancel{xv}-v=0$

$\Rightarrow x\frac{d^2v}{dx^2}+\frac{dv}{dx}=0$ - (2) again Put $\frac{dv}{dx}=Z$.

$\frac{d^2v}{dx^2}=\frac{dZ}{dx}$

(2) becomes $x\frac{dZ}{dx}+Z=0$

or $\frac{dZ}{Z}=-\frac{dx}{x}$ integrating $\log Z=-\log x+\log c$

$\therefore Z=\frac{c_1}{x}$ or, $\frac{dv}{dx}=\frac{c_1}{x}$

or $dv=c_1\frac{dx}{x}$ integrating

$v=c_1\log x+c_2$

$\therefore y=v.e^x=(c_1\log x+c_2)e^x$ is the required sol$^n$.