Notes on Differential eqn for Sem 2 Math Hons
Paper C-4T
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1(a) Lipschitz Condition: — A function $f(x,y)$ is said to satisfy a Lipschitz condition in a given region D in xy-plane if there exists a positive constant k such that

$$|f(x,y_2)-f(x,y_1)| \le k|y_2-y_1|$$

~~whether~~ whenever the points (x,y_1) and (x,y_2) both lie in D. The constant k is called a Lipschitz contant for the function $f(x,y)$.

b) If $f(x,y)$ satisfies the condition

$$\left|\frac{\partial f}{\partial y}\right| \le k$$

for all values of x, y in the given range D then for some constant k the Lipschitz condition is also satisfied.

(c) As a consequence of the definition, a function $f(x,y)$ satisfies Lipschitz condition if and only if there exists a constant $k>0$ such that

$$\frac{|f(x,y_2)-f(x,y_1)|}{|y_2-y_1|} \le k, \quad y_1 \ne y_2$$

whenever (x,y_1), (x,y_2) belong to D.

1. Example If S is defined by rectangle $|x| \le a$, $|y| \le b$. Show that the $f(x,y)=x^2+y^2$ satisfies the Lipschitz condition. Find the Lipschitz condition. $\left.\begin{matrix} y_1=b \\ y_2=b\end{matrix}\right\}$

Sol. Let (x,y_1) and (x,y_2) be two arbitrary points in the rectangle 'S'. Then we have $|f(x,y_2)-f(x,y_1)| = |\cancel{x^2+y^2}| \, |(x^2+y_2^2)-(x^2+y_1^2)|$

as $f(x,y)=x^2+y^2$; $= |y_2^2-y_1^2| = |(y_2+y_1)(y_2-y_1)|$

hence $|f(x,y_2)-f(x,y_1)| \le 2b(y_2-y_1)$ since $|y|<b$ in S

Showing the Lipschitz Condition is satisfied.

Here Lipschitz Constant is $k=2b$.

Ex. 2 Illustrate by an example that a continuous function may not satisfy a Lipschitz condition on a rectangle.

Solⁿ: Consider $f(x,y) = y^{2/3}$ on the rectangle $|x| \le 1$, $|y| \le 1$ - - ①

Clearly $f(x,y)$ is continuous in the rectangle S.

Here $\left|\frac{\partial}{\partial y} f(x,y)\right| = \frac{2}{3y^{1/3}} \to \infty$ when $y \to 0$. ②

Since $y = 0$ is a point of the rectangle S, ② shows that the Lipschitz condition is not satisfied by the function $f(x,y) = y^{2/3}$ on the rectangle S.

Ex ③ Show that $f(x,y) = 4x^2 + y^2$ on $R: |x| \le 1, |y| \le 1$ satisfy Lipschitz condition.

Solⁿ. Since $\left|\frac{\partial f}{\partial y}\right| = |2y| \le 2$ for $(x,y) \in R$.

Hence $f(x,y)$ satisfies Lipschitz condition with Lipschitz constant 2.

Picard's Theorem:—

Statement: — Let $f(x,y)$ be continuous in a domain D of the (x,y) plane and let M be a constant such that $|f(x,y)| \le M$ in D. ① Let $f(x,y)$ satisfy in D the Lipschitz condition in y namely - - ②

$$|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|$$

where the constant K is independent of x, y_1, y_2. Let the rectangle R, defined by $|x - x_0| \le h$, $|y - y_0| \le k$ (small) - - ③ lie in D where $Mh < k$. Then for $|x - x_0| \le h$, the differential equation $dy/dx = f(x,y)$ has a unique solution $y = y(x)$ for which $y(x_0) = y_0$.

Ex. Show that $f(x,y) = xy^2$ satisfies Lipschitz condition on the rectangle $R: |x| \le 1, |y| \le 1$ but not satisfy on the strip $S: |x| \le 1, |y| < \infty$

1st method: We have $|f(x,y_2) - f(x,y_1)| = |xy_2^2 - xy_1^2|$ as $f(x,y) = xy^2$

or $= |x||y_2 + y_1||y_2 - y_1|$ — Ⓐ in the given rectangle

$\le (1)(2)|y_2 - y_1|$ showing Lipschitz condition satisfied

Next $\left|\frac{f(x,y_2) - f(x,0)}{y_2 - 0}\right| = |x||y_2| \to \infty$ when $y_2 \to \infty$, if $|x| \ne 0$. Showing Lipschitz condition not satisfied.

Method 2: $\left|\frac{\partial f}{\partial y}\right| = 2|xy| = 2|x||y|$ in rectangle $|x| \le 1, |y| \le 1$

$\left|\frac{\partial f}{\partial y}\right| \le 2$ for each $(x,y) \in R$ showing Lipschitz condition satisfied with Lipschitz constant 2. On the other hand on the strip $S: |x| \le 1$, $|y| < \infty$, ~~show~~ $2|x||y|$ shows $\left|\frac{\partial f}{\partial y}\right|$ is unbounded when $y \to \infty$. Hence Lipschitz condition is not satisfied.

General solution of homogeneous differential equation:

An ordinary diff. Eqn. of nth order has the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_n y = R(x) \quad \cdots (1)$$

where $p_1, p_2 \cdots p_n$ are constants or function of x only. and dependent variable and derivatives appears only in the 1st degree and not multiplied together.

If $R(x) = 0$ then (1) will be called Homogeneous diff. eqn. for second order diff. eqn. $n = 2$ and hence 2nd order homogeneous diff. eqn. with constant coefficient will be

$$\frac{d^2y}{dx^2} + l\frac{dy}{dx} + ny = 0 \cdots (2)$$ l, n are constant.

it may be written in operator s.t. $D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$ so (2) may be written

$$D^2y + lDy + ny = 0 \Rightarrow (D^2 + lD + n)y = 0.$$

2nd order non homogeneous diff. eqn is

$$D^2y + lDy + ny = R(x) \quad \cdots (3)$$

The general soln. of (3) will have two parts

i.e $y = y_c + y_p$ where y_c is Complementary function (C.F) ~~yp is par~~ involves arbitrary constants and y_p is (P.I.) Particular integral does not involve any arbitrary Const.

In homogeneous eqn. C.F is general solution.

If in (3) $R(x) \neq 0$ then (3) will be non homogeneous eqn. its general solution has two parts C.F & P.I.

C.F. will be obtained ~~rega~~ reducing (3) to homogeneous eqn.

$y = e^{mx}$

then $Dy = me^{mx}$, $D^2y = m^2 e^{mx}$

then from $D^2y + lDy + ny = 0$ we get

$(m^2 + lm + n)e^{mx} = 0$ but $e^{mx} \neq 0$ So

$m^2 + lm + n = 0$ — (4) which is called A.E. (Auxiliary Equation)

on solving we get two roots of m.

(i) two roots different then the general sol^n (gl. Sol^n) (m_1, m_2)

is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where c_1, c_2 are arbitrary Consts.

(ii) when two ~~eq~~ roots are equal $m_1 = m_2$

The gl. Sol^n. of homogeneous diff. equ^n is

$y = (c_1 + c_2 x)e^{mx}$ when x is independent variable.

for 3rd order homogeneous diff. equ^n.

(a) ~~m ho~~ A.E. will be 3rd degree equ^n. $(m^3 + am^2 + bm + c) = 0$.

Then m will have 3 different or 3 equal values.

for different values. (m_1, m_2, m_3)

gl. sol^n is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

(b) where c_1, c_2, c_3 are arbitrary Constant

for equal values of m the gl. sol^n is $y = (c_1 + c_2 x + c_3 x^2)e^{mx}$

iii) If m will have one pair of complex roots $\alpha \pm i\beta$.

Then the gl. sol^n is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ or

$= c_1 e^{\alpha x} \sin(\beta x + c_2)$ or $c_1 e^{\alpha x} \cos(\beta x + c_2)$

(iv) one pair of surd roots $m = \alpha \pm \sqrt{\beta}$ then gl. sol^n.

is $y = e^{\alpha x}(c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})$ or $c_1 e^{\alpha x} \cosh(x\sqrt{\beta} + c_2)$

or $c_1 e^{\alpha x} \sinh(x\sqrt{\beta} + c_2)$

Examples

1. Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$.

Given equ^n. Can be written as

$(D^2 - 8D + 15)y = 0$.

For trial sol^n $y = e^{mx}$

the auxiliary eqn will be

$m^2-8m+15=0 \Rightarrow (m-3)(m-5)=0,\ m=3,5$ (different

So the required soln is $y=c_1e^{3x}+c_2e^{5x}$, c_1, c_2 arbitrary consts

2. Solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$ can be

written as $(D^2-6D+9)y=0$

its A.E is $m^2-6m+9=0 \Rightarrow (m-3)^2=0 \Rightarrow m=3,3$
equal roots

So the required soln is $y=(c_1+c_2x)e^{3x}$, c_1, c_2 arbitrary consts.

3. Solve $\frac{d^5y}{dx^5}-\frac{d^3y}{dx^3}=0 \Rightarrow (D^5-D^3)y=0$

So its A.E is $m^5-m^3=0 \Rightarrow m^3(m^2-1)=0$.

$m=0,0,0,1,-1$

general soln is $y=(c_1+c_2x+c_3x^2)e^{0x}+c_4e^{x}+c_5e^{-x}$

4. Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$ when $y=2$, $\frac{dy}{dx}=\frac{d^2y}{dx^2}$ when $x=0$.

Soln. Here the A.E is $m^2+4m+5=0$

or $m=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$

The complementary function is $y=e^{-2x}(A\cos x+B\sin x)$ — (I)

on putting $y=2$, $x=0 \Rightarrow 2=(A+0) \Rightarrow A=2$

Putting $A=2$ in (1) we get $y=e^{-2x}(2\cos x+B\sin x)$

on differentiating $\frac{dy}{dx}=e^{-2x}(-2\sin x+B\cos x)-\cancel{2e^{-2x}(2\cos x+B}$

$-2e^{-2x}(2\cos x+B\sin x)$

$=e^{-2x}[(-2B-2)\sin x+(B-4)\cos x]$

$\frac{d^2y}{dx^2}=e^{-2x}[(-4B+6)\cos x+(3B+8)\sin x]$ as $\frac{dy}{dx}=\frac{d^2y}{dx^2}$ when $x=0$.

So $B-4=-4B+6 \Rightarrow B=2$

hence gl. soln is $y=2e^{-2x}(\sin x+\cos x)$

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EX. Find the primitive of $(D^2-2D+5)^2 y = 0$.

Sol^n. Here A.E is $(D^2-2D+5)^2=0$ so that $m^2-2m+5=0$ ~~$D^2-2D+5=0$~~ (twice) and hence

$m = \left(\frac{2 \pm \sqrt{-16}}{2}\right) = 1 \pm 2i$ (twice).

Required sol^n is $y = e^x\{(c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x\}$.

c_1, c_2, c_3, c_4 being arbitrary constants

EX. Solve $(D^2+4)y=0$ given that $y=2$ and $\frac{dy}{dx}=0$ when $x=0$.

Sol^n. A.E is $m^2+4=0 \Rightarrow m = 0 \pm 2i$ do your self

EX. Solve $(D^4-m^4)y=0$.

Sol^n. Auxiliary equ^n. $D^4-m^4=0$ or $(D^2-m^2)(D^2+m^2)=0$

Hence $D = m, -m, \pm im$.

So the sol^n is $y = c_1e^{mx} + c_2e^{-mx} + e^{0x}(c_3\cos mx + c_4\sin mx)$

EX. Solve the initial value problem:

$(D^2-2KD+K^2)y=0$, $[y(0)=\sqrt{2}$, $y'(0)=K\sqrt{2}$, K is constant

$y(0)=\sqrt{2}$ means when $x=0$, $y=\sqrt{2}$ etc

Sol^n. ~~A.E is~~ $(D^2-2K+K^2)y=0$ its A.E is $m^2-2K+K^2=0$.

or $m = K, K$

The sol^n is $y=(c_1+c_2x)e^{Kx}$ — (1) Now putting $x=0$, $y=\sqrt{2}$ in (1)

we get $\sqrt{2} = (c_1+0)e^0 \Rightarrow c_1=\sqrt{2}$

again putting $c=\sqrt{2}$ we get $y=(\sqrt{2}+c_2x)e^{Kx}$

differentiating $\frac{dy}{dx} = e^{Kx}[K\sqrt{2}+c_2xK+c_2]$

Put $\frac{dy}{dx}=K\sqrt{2}$, when $x=0$, so ~~$y=K\sqrt{2}$~~

$K\sqrt{2} = e^0(K\sqrt{2}+0+c_2) \Rightarrow c_2=0$.

Hence the sol^n is $y=\sqrt{2}e^{Kx}$.