

Notes on Diff. Equⁿ. Sem-2 GEN Page 1
 DSC/BT (CC-2) (Euler's homogeneous linear equation)
 Teacher: Amalansu Sekhar Pattanayak.

An equation of the type $x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_{n-1} x \frac{dy}{dx} + p_n y = f(x)$... (1)

in which $p_1, p_2, \dots, p_n$ are constants is known as Euler's homogeneous linear equation or Cauchy-Euler equation. It may be transformed into linear equation with constant co-efficient by the substitution $x = e^z$ or $z = \log x$ so that $\frac{dx}{dz} = x$, or, $\frac{dz}{dx} = \frac{1}{x}$. (You may put $x = e^t, t = \log x$)

Exd solve $(x^2 D^2 - x D + 2)y = x \log x$... (2)
 Solⁿ. Let $x = e^z$ so that $z = \log x$ and let $\theta = \frac{d}{dz}$... (2)

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \therefore x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x D = \theta$$

$$\text{or, } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dz} \right)$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dz} \left(\frac{dy}{dz} \right) \cdot \frac{dz}{dx}$$

$$\therefore x^2 D^2 y = x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = (\theta^2 - \theta)y = \theta(\theta-1)y$$

[Similarly $x^3 D^3 y = \theta(\theta-1)(\theta-2)y$. $\theta = \frac{d}{dz}$]

Then the given equation (1) becomes
 $\{\theta(\theta-1) - \theta + 2\}y = z e^z \Rightarrow (\theta^2 - 2\theta + 2)y = z e^z$
 The Auxiliary equⁿ becomes $\theta^2 - 2\theta + 2 = 0$
 giving $\theta = 1 \pm i$

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$$\therefore C.F. = e^z (C_1 \cos z + C_2 \sin z) = x [C_1 \cos(\log x) + C_2 \sin(\log x)]$$

C_1, C_2 arbitrary Constant.

$$P.I. = \frac{1}{D^2 - 2D + 2} \cdot z e^z = e^z \frac{1}{(D+1)^2 - 2(D+1) + 2} \cdot z = e^z \cdot \frac{1}{D^2 + 1} z$$

$$= e^z (1 + D^2)^{-1} \cdot z = e^z (1 - D^2 + \dots) z$$

$$= e^z \cdot z$$

$$= x \log x \quad \left(\text{as } x = e^z, z = \log x \right)$$

$\left[\frac{d^2 z}{dz^2} = 0 \right]$

Required solⁿ is $y = x [C_1 \cos(\log x) + C_2 \sin(\log x)] + x \log x$.

Ex(2) Solve $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10 \left(x + \frac{1}{x} \right)$ $\rightarrow$ ①

Solⁿ: Put $x = e^t$ or $t = \log x$, $\Rightarrow \frac{dt}{dx} = \frac{1}{x}$, $\frac{dx}{dt} = x$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} = 1$$

$$x \frac{dy}{dx} = \frac{dy}{dt} = y \quad \text{where } \theta = \frac{d}{dt}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d}{dt} \left(\frac{dy}{dt} \right) \frac{dt}{dx}$$

$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}$$

$$\therefore x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt} = (\theta^2 - \theta)y \quad \text{as } \theta = \frac{d}{dt}$$

$$\text{So } x^3 \frac{d^3 y}{dx^3} = \theta(\theta-1)(\theta-2)y$$

Then the given equation becomes

$$\{ \theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2 \} y = 10(e^t + e^{-t})$$

$$\text{or, } (\theta^3 - \theta^2 + 2)y = (\theta+1)(\theta^2 - 2\theta + 2)y = 10(e^t + e^{-t})$$

Ex 3. A.E. $m^3 - m^2 + 2 = 0$ by trial set $y = e^{mt}$

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giving $m = -1, 1 \pm i$

C.F. is $C_1 e^{-t} + e^t (C_2 \cos t + C_3 \sin t)$

$$= \frac{C_1}{x} + x \{ C_2 \cos(\log x) + C_3 \sin(\log x) \}$$

$$\therefore P.I. = \frac{1}{\theta^3 - \theta^2 + 2} \cdot 10(e^t + e^{-t})$$

$$= \frac{10}{\theta^3 - \theta^2 + 2} \cdot e^t + \frac{10}{\theta^3 - \theta^2 + 2} \cdot e^{-t}$$

$$= \frac{10}{2} \cdot e^t + 10 \cdot e^{-t} \frac{1}{(\theta-1)^3 - (\theta-1)^2 + 2}$$

$$= 5e^t + 10 \cdot e^{-t} \frac{1}{\theta^3 - 4\theta^2 + 5\theta}$$

$$= 5e^t + 10 \cdot e^{-t} \frac{1}{5\theta \left(1 + \frac{\theta^2 - 4\theta}{5}\right)}$$

$$= 5e^t + 10 \cdot e^{-t} \cdot \frac{1}{5\theta} (1 + \dots)$$

$$= 5e^t + 10e^{-t} \cdot \frac{1}{5} \cdot t = 5e^t + 2e^{-t} \cdot t$$

$$= 5x + 2 \cdot \frac{1}{x} \log x$$

Hence complete solution is

$$y = \frac{C_1}{x} + x(C_2 \cos \log x + C_3 \sin \log x) + 5x + \frac{2}{x} \log x$$

EX(3) Find the values of λ for which all solutions of $x^2 \left(\frac{d^2 y}{dx^2} \right) - 3x \frac{dy}{dx} - \lambda y = 0$ tend $\rightarrow$ Zero, $x \rightarrow \infty$.

$$\text{Sol: } (x^2 D^2 - 3x D + \lambda) y = 0 \quad \text{--- (1)} \quad D = \frac{d}{dx}$$

Put $x = e^z$ so that $z = \log x$ let $\theta = \frac{d}{dz}$

Then $x D = \theta$ and $x \frac{dy}{dx} = \theta y$ and $x^2 \frac{d^2 y}{dx^2} = \theta(\theta-1)y$

The given equation (1) becomes $\{ \theta(\theta-1) + 3\theta - \lambda \} y = 0$

$$\text{or } (\theta^2 - 2\theta - \lambda) y = 0 \quad \text{--- (2)}$$

Auxiliary equation $\theta^2 - 2\theta - \lambda = 0$ giving

$$\theta = \frac{2 \pm (4 + 4\lambda)^{1/2}}{2} = -1 \pm (1 + \lambda)^{1/2} \text{ where } \lambda > -1$$

Hence the general solution is $y = C_1 e^{-\{1 - (1 + \lambda)^{1/2}\}x} + C_2 e^{-\{1 + (1 + \lambda)^{1/2}\}x}$

$$y = C_1 x^{-[1 - (1 + \lambda)^{1/2}]} + C_2 x^{-[1 + (1 + \lambda)^{1/2}]} \text{ using } x = e^z$$

Since all solutions must tend to zero as $x \rightarrow \infty$, λ must be chosen to satisfy the following condition $1 - (1 + \lambda)^{1/2} > 0$

$$\text{or } (1 + \lambda)^{1/2} < 1 \text{ so that } \lambda < 0.$$

hence $-1 \leq \lambda < 0$ is the required values of λ .

EX: Solve $(1 + 2x)^2 \frac{d^2 y}{dx^2} - 8(1 + 2x) \frac{dy}{dx} + 16y = 8(1 + 2x)^2$ [Legendre's equation]

Sol: Let us put $1 + 2x = e^z$ i.e. $z = \log(1 + 2x)$ Then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{1 + 2x} \cdot 2 \cdot \frac{dy}{dz} \Rightarrow (1 + 2x) \frac{dy}{dx} = 2 \frac{dy}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left\{ \left(\frac{2}{1 + 2x} \right) \frac{dy}{dz} \right\} = \frac{2}{1 + 2x} \cdot \frac{d}{dz} \left(\frac{dy}{dz} \right) \cdot \frac{dz}{dx} = \frac{4}{(1 + 2x)^2} \cdot \frac{d^2 y}{dz^2}$$

$$= \frac{4}{(1 + 2x)^2} \cdot \frac{d^2 y}{dz^2} = \frac{4}{(1 + 2x)^2} \cdot \frac{d^2 y}{dz^2}$$

$$(1 + 2x)^2 \frac{d^2 y}{dx^2} = 4 \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) = 4(D^2 - D)y \quad [Dy = \frac{dy}{dz}]$$

The given eqn. becomes $4 \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) - 16 \frac{dy}{dz} + 16y = 8e^{2z}$

$$\text{or, } \frac{d^2 y}{dz^2} - 4 \frac{dy}{dz} + 4y = 2e^{2z}$$

for C.F. the A.E. is $D^2 - 4D + 4 = 0$ i.e. $m^2 - 4m + 4 = 0$

$$\text{i.e. } (m - 2)^2 = 0 \Rightarrow m = 2, 2$$

C.F. is $(C_1 + C_2 z)e^{2z} = \{C_1 + C_2 \log(1 + 2x)\} (1 + 2x)^2$

$$P.I. = \frac{1}{(D - 2)^2} 2e^{2z} = e^{2z} \cdot \frac{1}{(D + 2 - 2)^2} \cdot 2 = e^{2z} \cdot \frac{1}{D^2} \cdot 2 = \frac{2}{2^2} \cdot e^{2z} = \frac{1}{2} e^{2z}$$

The general solⁿ is $y = (C_1 + C_2 z)e^{2z} + \frac{1}{2} e^{2z}$

$$\therefore y = \{C_1 + C_2 \log(1 + 2x)\} (1 + 2x)^2 + (1 + 2x)^2 \left\{ \log(1 + 2x) \right\}$$

$$y = (1 + 2x)^2 \left[C_1 + C_2 \log(1 + 2x) + \left\{ \log(1 + 2x) \right\} \right] \text{ is the required solution.}$$