Notes on Diff. Eqn. Sem-2 Math Hons Page-1

Paper C4T (Euler's homogeneous linear equation)

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An equation of the type $x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}}$

$+ \cdots + p_{n-1} x \frac{dy}{dx} + p_n y = f(x)$ -- (1)

in which $p_1, p_2 \cdots p_n$ are constants is known as Euler's homogeneous linear equation or Cauchy-Euler equation. It may be transformed into linear equation with constan co-efficient by the substitution $x = e^z$ or $z = \log x$ so that

$\frac{dx}{dz} = x$, or, $\frac{dz}{dx} = \frac{1}{x}$. (~~some~~ you may put $x = e^t$, $t = \log x$)

for second order $n=2$

EX(1) Solve $(x^2D^2 - xD + 2)y = x\log x$ —— (1)

Soln. Let $x = e^z$ so that $z = \log x$ and let $\theta = \frac{d}{dz}$ ... (2)

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \therefore x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow xD = \theta$$

$D = \frac{d}{dx}$, $\theta = \frac{d}{dz}$

$$\text{or, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\therefore x^2D^2y = x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (\theta^2 - \theta)y = \theta(\theta-1)y$$

[Similarly $x^3D^3y = (\theta)(\theta-1)(\theta-2)y$]. $\theta = \frac{d}{dz}$

Then the given equation (1) becomes

$$\{\theta(\theta-1) - \theta + 2\}y = ze^z \Rightarrow (\theta^2 - 2\theta + 2)y = ze^z$$

The Auxiliary eqn. becomes $\theta^2 - 2\theta + 2 = 0$

giving $\theta = 1 \pm i$

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$\therefore$ C.F $= e^{z}(c_1\cos z + c_2\sin z) = x[c_1\cos(\log x) + c_2\sin\log x]$

c_1, c_2 arbitrary Constant.

$$P.I = \frac{1}{\theta^2-2\theta+2}\cdot z e^{z} = e^{z}\frac{1}{(\theta+1)^2-2(\theta+1)+2}\cdot z = e^{z}\cdot\frac{1}{\theta^2+1}z$$

$$= e^{z}(1+\theta^2)^{-1}\cdot z = e^{z}(1-\theta^2+\cdots)z \qquad \left[\frac{d^2z}{dz^2}=0\right.$$

$$= e^{z}\cdot z$$

$$= x\log x \qquad (\text{as } x=e^{z},\ z=\log x)$$

Required sol^n is $y = x[c_1\cos(\log x) + c_2\sin(\log x)] + x\log x$.

Ex(2) Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right) \longrightarrow (1)$

Sol^n: Put $x=e^{t}$ or $t=\log x$, $\Rightarrow \frac{dt}{dx}=\frac{1}{x}$, $\frac{dx}{dt}=x$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$x\frac{dy}{dx} = \frac{dy}{dt} = \theta y \qquad \text{where } \theta=\frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\cdot\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = (\theta^2-\theta)y \qquad \text{as } \theta=\frac{d}{dt}.$$

So $x^3\frac{d^3y}{dx^3} = \theta(\theta-1)(\theta-2)y$.

Then the given equation (1) becomes

$$\{\theta(\theta-1)(\theta-2)+2\theta(\theta-1)+2\}y = 10(e^{t}+e^{-t})$$

or, $(\theta^3-\theta^2+2)y = (\theta+1)(\theta^2-2\theta+2)y = 10(e^{t}+e^{-t})$

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~~C.F~~ . A.E $m^3 - m^2 + 2 = 0$. ~~$m^2 - 2m + 2 = 0$~~ by trial soln $y = e^{mt}$

giving $m = -1, 1 \pm i$

C.F is $c_1 e^{-t} + e^t(c_2 \cos t + c_3 \sin t)$

$= \frac{c_1}{x} + x\{c_2 \cos(\log x) + c_3 \sin(\log x)\}$

$\therefore$ $P.I = \frac{1}{\theta^3 - \theta^2 + 2} \cdot 10(e^t + e^{-t})$

$= \frac{10}{\theta^3 - \theta^2 + 2} \cdot e^t + \frac{10}{\theta^3 - \theta^2 + 2} \cdot e^{-t}$.

$= \frac{10}{2} \cdot e^t + 10 \cdot e^{-t} \frac{1}{(\theta-1)^3 - (\theta-1)^2 + 2} \cdot 1$

$= 5 \cdot e^t + 10 \cdot e^{-t} \frac{1}{\theta^3 - 4\theta^2 + 5\theta} \cdot 1$

$= 5e^t + 10 \cdot e^{-t} \frac{1}{5\theta}\left(1 + \frac{\theta^2 - 4\theta}{5}\right)^{-1} \cdot 1$

$= 5e^t + 10 \cdot e^{-t} \cdot \frac{1}{5\theta}(1 + \cdots) \cdot 1$

$= 5e^t + 10 e^{-t} \cdot \frac{1}{5} \cdot t$ $= 5e^t + 2e^{-t} \cdot t$.

$= 5x + 2 \cdot \frac{1}{x} \log x$

Hence complete solution is

$y = \frac{c_1}{x} + x(c_2 \cos \log x + c_3 \sin \log x) + 5x + \frac{2}{x} \log x$

EX 3 Find the values of λ for which all solutions of ~~$x^2\left(\frac{d^2y}{dx^2}\right) - 3x\frac{dy}{dx}$~~ tend $\to$ Zero, $x \to \infty$.

$x^2\left(\frac{d^2y}{dx^2}\right) - 3x\frac{dy}{dx} - \lambda y = 0$.

Sol^n: $(x^2D^2 - 3xD + \lambda)y = 0$ ...(1), $D = \frac{d}{dx}$

put $x = e^z$ so that $z = \log x$ let $\theta = \frac{d}{dz}$.

Then $xD = \theta$ i.e. $x\frac{dy}{dx} = \theta y$ and $x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$.

The given equation (1) becomes $\{\theta(\theta-1) + 3\cdot\theta, -\lambda\}y = 0$

or $(\theta^2 - 2\theta - \lambda)y = 0$. ——— (2)

Auxiliary equation $\theta^2 - 2\theta - \lambda = 0$ giving

$$\theta = \frac{-2 \pm (4+4\lambda)^{1/2}}{2} = -1 \pm (1+\lambda)^{\frac{1}{2}} \quad \text{where } \lambda \geq -1$$

Hence the general solution is $y = c_1 e^{-\{1-(1+\lambda)^{\frac{1}{2}}\}z} + c_2 e^{-\{1+(1+\lambda)^{\frac{1}{2}}\}z}$

$$y = c_1 x^{-[1-(1+\lambda)^{\frac{1}{2}}]} + c_2 x^{-[1+(1+\lambda)^{\frac{1}{2}}]} \quad \text{using } z = \log x$$

Since all solutions must tend to zero as $x \to \infty$, λ must be chosen to satisfy the following condition $1-(1+\lambda)^{1/2} > 0$

or $(1+\lambda)^{1/2} < 1$ so that $\lambda < 0$.

hence $-1 \leq \lambda < 0$ is the required values of λ.

EX: Solve $(1+2x)^2 \frac{d^2y}{dx^2} - 8(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$ [Legendre's equation]

Sol^n: Let us put $1+2x = e^z$ ie $z = \log(1+2x)$ Then

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{1+2x}\cdot 2\cdot\frac{dy}{dz} \Rightarrow (1+2x)\frac{dy}{dx} = 2\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\left(\frac{2}{1+2x}\right)\frac{dy}{dz}\right\} = \frac{2}{1+2x}\cdot\frac{d}{dz}\frac{dy}{dz}\cdot\frac{dz}{dx} - \frac{4}{(1+2x)^2}\cdot\frac{dy}{dz}$$

$$= \frac{2}{1+2x}\cdot\frac{d^2y}{dz^2}\cdot\frac{2}{1+2x} - \frac{4}{(1+2x)^2}\cdot\frac{dy}{dz}$$

$$(1+2x)^2\frac{d^2y}{dx^2} = 4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = 4(\theta^2 - \theta) \quad \left[\theta y = \frac{dy}{dz}\right]$$

The given equ^n becomes $4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 16\frac{dy}{dz} + 16y = 8e^{2z}$.

or, $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 2e^{2z}$

for C.F the A.E is $D^2 - 4D + 4 = 0$ ie $m^2 - 4m + 4 = 0$

ie $(m-2)^2 = 0$ $\therefore m = 2, 2$

C.F is $(c_1 + c_2 z)e^{2z} = \{c_1 + c_2\log(1+2x)\}(1+2x)^2$

$$P.I = \frac{1}{(D-2)^2}\cdot 2e^{2z} = e^{2z}\cdot\frac{1}{(D+2-2)^2}\cdot 2 = e^{2z}\cdot\frac{1}{D^2}\cdot 2 = e^{2z}\cdot z^2$$

The general sol^n is $y = (c_1 + c_2 z)e^{2z} + e^{2z}\cdot z^2$

$$\therefore y = \{c_1 + c_2\log(1+2x)\}\cdot(1+2x)^2 + (1+2x)^2\{\log(1+2x)\}^2$$

$$y = (1+2x)^2\left[c_1 + c_2\log(1+2x) + \{\log(1+2x)\}^2\right]$$ is the required solution.

Notes on

Math Dept.

Solution of D.E (Diff. Equ^n) by method of undermined coefficents for Sem 2 Math Hons. Paper C-4T. (unit-1)

To solve the D.E. of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x) \ldots (1)$$ where P, Q Constants.

Complete sol^n $= y_c + y_p = C.F + P.I.$

for c.f making $R(x)=0$. Now there are Some rules to find the y_p.

The method of undetermined co-efficients is a Procedure to find y_p when $R(x)$ is an exponential, a Sine or Cosine a Polynomial or Some Combination of such.

Rule (I) when the R.H.S. $R(x)$ is an exponential i.e $R = e^{ax}$

a) when e^{ax} is not in c.f. then trial sol^n is $y_p = Ae^{ax}$ where A is undertimed to be found.

b) when e^{ax} occurs in C.F then we assume $y_p = Axe^{ax}$

c) when a double root i.e C.F contains e^{ax} and xe^{ax} then we assume Ax^2e^{ax}

Rule (II) when R.H.S of (1) or $R(x)$ is a Polynomial or constant that is $R(x) = a_0 + a_1x + \ldots a_nx^n$.

Take the $y_p = A_0 + A_1x + A_2x^2 + \ldots A_nx^n$

in Particular if $R(x) = a_0$ take $y_p = A_0$.

Rule (III) when $R(x)$ Contains $\sin ax$ or $\cos ax$

a) when $\sin ax$ or $\cos ax$ not in C.F. we take then $y_p = A\sin ax + B\cos ax$.

b) when $\sin ax$ or $\cos ax$ is in C.F. then trial solution is $y_p = x(A\sin ax + B\cos ax)$ A, B to be found.

Rule (IV) when $R(x)$ Contain $e^{ax}\sin bx$ or $e^{ax}(a_0 + a_1x + \ldots)$ y_p will be taken accordingly with the help of Rule (I), (II) & (III)

Consult with the Book Differential Equ^n by
1) R.H Ghosh and Maity 2) by Dr. M.D. Raisinghania

Some solution of problems.

Ex (1) Solve using method of undermined co-efficients the equation.

$$\frac{d^2y}{dx^2}+10\frac{dy}{dx}+25y=14e^{-5x} \quad \text{—— (1)}$$

Sol^n: the equ^n is $(D^2+10D+25)y=14.e^{-5x}$

A.E is $m^2+10m+25=0$ [for C.F we take $R(x)=0$.

$(m+5)^2=0$, $\Rightarrow$ $m=-5,-5$ (~~double roots~~

~~yp = e~~ . $C.F=(c_1+c_2x)e^{-5x}$

e^{-5x} is a double root ~~of~~ R.H.S in C.F.

So $y_p=Ax^2e^{-5x}$ —— (2)

Now $D^2y_p+10Dy_p+25y_p=14e^{-5x}$ ⟶ (3)

from (2) $Dy_p=A\left(x^2\times -5e^{-5x}+2x.e^{-5x}\right)$

$=Ae^{-5x}(2x-5x^2)$

$D^2y_p=A\left\{e^{-5x}(2-10x)-5e^{-5x}(2x-5x^2)\right\}$

$=Ae^{-5x}(2-20x+25x^2)$

Putting the values of Dy_p, D^2y_p and y_p in (3) we get

$Ae^{-5x}(2-20x+25x^2)+10(Ae^{-5x})(2x-5x^2)+25Ax^2e^{-5x}=14e^{-5x}$

$Ae^{-5x}(2-20x+25x^2+20x-50x^2+25x^2)=14e^{-5x}$

or $2Ae^{-5x}=14e^{-5x}$ $\Rightarrow$ $A=14/2$ $\Rightarrow$ $A=7$.

$\therefore y_p=7x^2e^{-5x}$

Thus C.P or Complete Primitive is

$$y=(c_1+c_2x)e^{-5x}+7x^2e^{-3x}$$

Ex (2) Solve by method of undetermined co-efficients.

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}=x+e^x\sin x$$

For C.F. ~~Complementary function~~ Auxiliary Equation of $(D^2-3D)y=0$

is $m^2 - 3m = 0 \Rightarrow m = 0,\ m = 3$

$\therefore CF = c_1 + c_2 e^{3x}$

To find P.I ~~(yp)~~ take

$y_p = x(A + Bx) + e^x(C\sin x + D\cos x)$; As y term absent in the D.E

$\therefore D(y_p) = A + 2Bx + e^x(C\cos x - D\sin x) + e^x(C\sin x + D\cos x)$

$D^2(y_p) = 2B + e^x(-C\sin x - D\cos x) + e^x(C\cos x - D\sin x)$
$\quad + e^x(C\cos x - D\sin x) + e^x(C\sin x + D\cos x)$

Putting the values of $D^2 y_p$, Dy_p and y_p in

$D^2 y_p - 3Dy_p = x + e^x \sin x$

$2B + e^x(2C\cos x - 2D\sin x) - 3A - 6Bx - 3e^x\{(C+D)\cos x + (C-D)\sin x\}$
$= x + e^x \sin x$

Comparing the Co-efficient and Constant term of both sides

$2B - 3A = 0$; $-6B = 1$
$\therefore 3A = 2B$ $\quad B = -\frac{1}{6}$
$3A = 2 \times -\frac{1}{6} = -\frac{1}{3}$
$A = -\frac{1}{9}$

Coefficient of $e^x \sin x$, $e^x \cos x$

~~$-2D - C - D = 1$~~

$-2D - 3C + 3D = 1$
$D - 3C = 1$

and $2C - 3C - 3D = 0$
$-C - 3D = 0 \Rightarrow$ ~~$C = 3$~~ $C = -3D$

~~$D + 9D$~~ ~~$4D$~~

$D = \frac{1}{10}$.

and $C = -\frac{3}{10}$

Hence the Complete sol^n is $y = y_c + y_p$.

$\therefore y = (c_1 + c_2 e^{3x}) - \frac{1}{9}x - \frac{1}{6}x^2 + e^x\left(-\frac{3}{10}\sin x + \frac{1}{10}\cos x\right)$

Ex Solve by the method of undetermined co-efficients

$$(D^2-4D+4)y = x^3e^{2x} + xe^{2x} = e^{2x}(x^3+x)$$

Solⁿ: For C.F, the auxiliary equation is $m^2-4m+4=0 \Rightarrow m=2,2$

hence $C.F = (c_1+c_2x)e^{2x}$, c_1, c_2 arbitrary constant

Now as e^{2x} is in C.F. and double roots

so $y_p = x^2e^{2x}(Ax^3+Bx^2+Cx+E)$, Here A, B, C, E arbitrary constant.

find $Dy_p = 2x\,e^{2x}(Ax^3+Bx^2+Cx+E) + x^2\cdot 2e^{2x}\cdot(Ax^3+Bx^2+Cx+E)$
$+ x^2e^{2x}(3Ax^2+2Bx+C)$

$D^2y_p = 2\{e^{2x}(Ax^3+Bx^2+Cx+E) + x\cdot 2e^{2x}(Ax^3+Bx^2+Cx+E)$
$+ x\cdot e^{2x}(3Ax^2+2Bx+C)\}$
$+2\{2x\,e^{2x}(Ax^3+Bx^2+Cx+E) + x^2\cdot 2e^{2x}(Ax^3+Bx^2+Cx+E)$
$+x^2e^{2x}(3Ax^2+2Bx+C)\} + 2x\,e^{2x}(3Ax^2+2Bx+C)$
$+x^2\cdot 2e^{2x}(3Ax^2+2Bx+C) + x^2e^{2x}(6Ax+2B)$

Putting the values in $D^2y_p - 4Dy_p + 4y_p = e^{2x}(x^3+x)$

Comparing the coefficients $e^{2x}x$, $e^{2x}x^2$, $e^{2x}x^3$, x^4e^{2x} etc

and solving $A=\frac{1}{20}$, $B=0$, $C=\frac{1}{6}$, $E=0$

$\therefore y_p = \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$.

General solⁿ is $y = e^{2x}(c_1+c_2x) + \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$.