Dept. of Math.

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DSC-1B (CC-2) Diff. Equation

: Exact diff. Equⁿ; : Integrating factor

: Equation solvable for x, y, p.

1. The necessary and sufficient condition for the D.E. (Differential Equation) $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Ex 1 Verify exactness of the D.E

$$(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$$

Solⁿ: Comparing with $Mdx + Ndy = 0$ Here

$$M = x^2 - 4xy - 2y^2; \quad N = y^2 - 4xy - 2x^2$$

$\therefore \frac{\partial M}{\partial y} = -4x - 4y$ and $\frac{\partial N}{\partial x} = -4y - 4x$

so that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the given D.E is exact

Ex 2. Examine whether $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$ is exact D. Equation.

Solⁿ: Let $M = \cos y + y\cos x$ & $N = \sin x - x\sin y$

Then $\frac{\partial M}{\partial y} = -\sin y + \cos x$ and $\frac{\partial N}{\partial x} = \cos x - \sin y$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so this D.E is exact.

Exercise Examine whether (i) $(x^2+y^2+4)x\,dx + (x^2-y^2+9)y\,dy = 0$ ~~is exa~~ (ii) $(x^2+y^2+x)dx + xy = 0$

(iii) $(1+e^{x/y})dx + e^{x/y}\{1-(x/y)\}dy = 0$. are exact?

The students should know the following diff. equation are exact and it is known by inspection.

i) $x\,dy + y\,dx = d(xy)$

ii) $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$; iii) $\frac{x\,dy - y\,dx}{x^2} = d(y/x)$.

(iv) $\frac{y\,dx - x\,dy}{x^2+y^2} = d\{\tan^{-1}(\frac{x}{y})\}$

(v) $\frac{x\,dy - y\,dx}{xy} = d\{\log(\frac{x}{y})\}$ etc.

~~Consult~~ Consult with Diff Equation by Ghosh & Maity.

Rule (1) If the Differential Equation $M\,dx + N\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor (I.F.) of the given D.E.

EX (1). Find I.F of $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Equ^n (1) is homogeneous, but not exact ... (1).
(both M and N are homogeneous.)

and $Mx + Ny = x(x^2y - 2xy^2) - y(x^3 - 3x^2y)$
$= x^2y^2 \neq 0$.

So. I.F (Integrating factor) of the D.E (1) is $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

EX. (2) $x^2y\,dx - (x^3 - y^3)dy = 0$ find Integrating factor. Do yourself.

Rule ② for finding integrating factor

If the equation $Mdx + Ndy = 0$ is of the form $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$, then its I.F is $\frac{1}{Mx - Ny}$ provided $Mx - Ny \neq 0$.

Ex ① Find I.F. of $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$ ... ①

Comparing with the above Rule ②

Here $Mx - Ny = xy(xy\sin xy + \cos xy) - xy(xy\sin xy - \cos xy) = 2xy\cos xy \neq 0$.

Hence required I.F $= \frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy}$

Exercise ② Find the I.F. of $y(1+xy)\,dx + x(1-xy)\,dy = 0$

Rule ③ If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$ then $e^{\int f(x)dx}$ is an integrating factor of $Mdx + Ndy = 0$.

Ex ① Find I.F of the D.E. $(x^2+y^2+x)dx + xy = 0$

Sol^n Comparing with $Mdx + Ndy = 0$

Here $M = x^2+y^2+x$, $N = xy$; $\frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = y$

it is not exact. Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - y) = \frac{1}{x} = f(x)$

So. I.F $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$.

Ex ② Find I.F. of $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)dy=0$.

Sol^n: Here $M=y+\frac{y^3}{3}+\frac{x^2}{2}$, $N=\frac{1}{4}(x+xy^2)$

$\frac{\partial M}{\partial y}=1+y^2$, $\frac{\partial N}{\partial x}=\frac{1}{4}(1+y^2)$ not exact

but $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{4}{x(1+y^2)}\left\{(1+y^2)-\frac{1}{4}(1+y^2)\right\}$

$=\frac{4}{x(1+y^2)}\times\frac{3}{4}(1+y^2)=\frac{3}{x}$

Hence I.F is $~~e^{\int\frac{3}{x}dx}~~=e^{\int\frac{3}{x}dx}=e^{3\log x}=e^{\log x^3}$

$=x^3$

Rule ④ If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$ then its I.F is $e^{\int f(y)dy}$.

Ex ① Find I.F of $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$.

Sol^n: Comparing with the Rule ④ here $M=xy^2-x^2$

and $N=3x^2y^2+x^2y-2x^3+y^2$, Now $\frac{\partial M}{\partial y}=2xy$

$\frac{\partial N}{\partial x}=6xy^2+2xy-6x^2$

$\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{xy^2-x^2}\left\{(6xy^2+2xy-6x^2)-2xy\right\}$

$=\frac{6x(y^2-x)}{x(y^2-x)}=6$.

Hence the I.F $=e^{\int 6dy}=e^{6y}$.

Ex ② Find I.F. of $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$.

③ Find I.F. of $(2xy^2-2y)dx+(3x^2y-4x)dy=0$.

Diff. equation

The differential of the type $\left(\frac{dy}{dx}\right)^2 + P_1\frac{dy}{dx} + P_2 = 0$ is called 1st order but not 1st degree.

1. ~~Solution of~~ Solve D.E $P^2 + P - 6 = 0$. where $P = \frac{dy}{dx}$

Sol^n. Here $P^2 + P - 6 = 0$ ---(1)

or $(P+3)(P-2) = 0$ Either $P = -3$ or $P = 2$ hence as $P = \frac{dy}{dx}$

so $\frac{dy}{dx} = -3$, or $dy = -3dx$

integrating $\int dy = -3\int dx \Rightarrow y = -3x + c_1$ ie $y + 3x - c_1 = 0$ ---(2)

where c_1 is an arbitrary Constant

again $P = 2 \Rightarrow \frac{dy}{dx} = 2 \Rightarrow dy = 2dx$

integrating $y = 2x + c_2 \Rightarrow y - 2x - c_2 = 0$ ---(3)

The general sol^n of the (1) is

$(y + 3x - c)(y - 2x - c) = 0$ here will be

The D.E. ~~is~~ of 1st order so one arbitrary Constant.

(2) Solve $xy\left(\frac{dy}{dx}\right)^2 - (x^2 - y^2)\frac{dy}{dx} - xy = 0$. writing $P = \frac{dy}{dx}$

then the D.Equation becomes $xyP^2 - (x^2 - y^2)P - xy = 0$

or, $xyP^2 - x^2P + y^2P - xy = 0 \Rightarrow xP(yP - x) + y(yP - x) = 0$

$\Rightarrow (yP - x)(xP + y) = 0$ hence $yP - x = 0$ gives

$yP = x \Rightarrow y\frac{dy}{dx} = x \Rightarrow ydy = xdx$ integrating

$\frac{y^2}{2} - \frac{x^2}{2} = c_1 \Rightarrow y^2 - x^2 = 2c_1 = c$

or, $y^2 - x^2 - c = 0$ ---(2) c_1 is arbitrary Const.

again $xP + y = 0$ gives $x\frac{dy}{dx} = -y \Rightarrow \frac{dx}{x} = -\frac{dy}{y}$ integratio

$\log|x| + \log|y| = \log c_2 \Rightarrow xy = c_2 \Rightarrow xy - c_2 = 0$ ---(3)

Hence the general soln from (2) & (3) becomes

$(y^2 - x^2 - c)(xy - c) = 0$.

(3) Equation involving no y or no x :

~~Solve~~ i.e. equation solving for x or y.

Ex(1) Solve (1) $y = 2px + y^2p^3$ ... (1) where $p = \frac{dy}{dx}$

Sol^n: solving for x i.e. $x = \frac{y}{2p} - \frac{y^2p^3}{2p}$.

i.e. $x = \frac{y}{2p} - \frac{y^2p^2}{2}$ ... (2) Differentiating w.r.to y

$\cancel{\frac{dx}{dy} = \frac{1}{2p} - \cdots}$

$$\frac{d}{dy}(x) = \frac{d}{dy}\left(\frac{y}{2p}\right) - \frac{d}{dy}\left(\frac{y^2p^2}{2}\right) \qquad \frac{dx}{dy} = \frac{1}{p}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2}\frac{p.1 - y.\frac{dp}{dy}}{p^2} - \frac{1}{2}\left[2yp^2 + 2p\frac{dp}{dy}.y^2\right]$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2}\left[\frac{p}{p^2} - \frac{y}{p^2}\frac{dp}{dy}\right] - p^2y - y^2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} - \frac{1}{2p} + \frac{y\,dp/dy}{2p^2} + p^2y + y^2p\frac{dp}{dy} = 0$$

or. $$\frac{1}{2p} + \frac{dp}{dy}\left(\frac{y}{2p^2} + y^2p\right) + p^2y = 0.$$

or $$\frac{1}{2p} + p^2y + \frac{dp}{dy}\left(\frac{y}{2p^2} + y^2p\right) = 0$$

or $$p\left(py + \frac{1}{2p^2}\right) + \frac{dp}{dy}y\left(py + \frac{1}{2p^2}\right) = 0$$

or, $\left(py + \frac{1}{2p^2}\right)\left(p + y\frac{dp}{dy}\right) = 0$ neglecting 1st factor as it has no $\frac{dp}{dy}$

hence $p + y\frac{dp}{dy} = 0 \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$

~~Dividing by $p\,dy$~~ [multiplying ~~by~~ dy and then dividing by py (both sides)] Now integrating

$\log p + \log y = \log c \Rightarrow py = c$ [c is arbitrary const.

or $p = \frac{c}{y}$ putting the value $p = \frac{c}{y}$ in equ^n (1)

The required solution is

$$y = 2x.\frac{c}{y} + y^2\left(\frac{c}{y}\right)^3 \text{ or, } y^2 = 2cx + c^3$$

Ex ② Solve $y = 2px + p^2 y \quad \left(p = \frac{dy}{dx}\right)$.

Solution: solving for x

$$2x = \frac{y}{p} - py = -py + \frac{y}{p} \quad \text{--- (I)}$$

Differentiating w.r.t y $\quad \left[\frac{1}{p} = \frac{dx}{dy}\right]$

$$2\frac{dx}{dy} = \frac{2}{p} = -p - y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

or, $\frac{2}{p} + p - \frac{1}{p} = -y\left(\frac{dp}{dy}\right) - \frac{y}{p^2}\frac{dp}{dy}$

or, $p + \frac{1}{p} = -y\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right)$

or. $p\left(1 + \frac{1}{p^2}\right) + y\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right) = 0$

or, $\left(1 + \frac{1}{p^2}\right)\left(p + y\frac{dp}{dy}\right) = 0$ neglecting $\left(1 + \frac{1}{p^2}\right)$ as it has no $\frac{dp}{dy}$.

$\therefore \quad p + y\frac{dp}{dy} = 0 \Rightarrow p\,dy + y\,dp = 0$

or, $\frac{dy}{y} + \frac{dp}{p} = 0 \Rightarrow \log(py) = \log c$

$\Rightarrow py = c \Rightarrow p = \frac{c}{y}$.

eliminating p from Equ^n ① by putting $p = \frac{c}{y}$

we get $2x = -\frac{c}{y} \times y + y \times \frac{y}{c} = -c + \frac{y^2}{c}$

or. $2xc - y^2 + c^2 = 0$ is the required solution.

Diff. equation ~~for~~ solvable for y.

Ex: ① Solve $y + px = x^4 p^2$ where $p = \frac{dy}{dx}$ --- ①

Sol^n: ~~for~~ Solving for $y \Rightarrow y = x^4p^2 - px$ --- ②

differentiating with respect to x and writing $\frac{dy}{dx} = p$ we get

from ② $p = 4x^3p^2 + 2x^4p\frac{dp}{dx} - \left[p + x\left(\frac{dp}{dx}\right)\right]$

$p + p - 4x^3p^2 + x\frac{dp}{dx} - 2x^4p\frac{dp}{dx} = 0.$

or $2p - 4x^3p^2 + \frac{dp}{dx}(x - 2x^4p) = 0$

or, $2p(1 - 2x^3p) + x\frac{dp}{dx}(1 - 2x^3p) = 0$

or, $(1 - 2x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$

neglecting 1st factor which has no $\frac{dp}{dx}$ term.

So $2p + x\frac{dp}{dx} = 0 \Rightarrow \frac{1}{p}dp + 2.\frac{1}{x}dx = 0$

integrating $\log p + 2\log x = \log c$

or $px^2 = c$ or $p = \frac{c}{x^2}$ (c is arbitrary constant)

Putting $p = \frac{c}{x^2}$ in $y + px = x^4p^2$ - ①

we get $y + \frac{c}{x^2}.x = x^4 \times \frac{c^2}{x^4}$

or, $xy + c = c^2$ is the required Sol^n.