Semester- IV

Course Type- core-8

Course Title - C8T: sequence of function

Topic: Examples of sequence of function

References: S.K. Mapa Book.

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$$|f_n(x)| \le K_1 \quad \& \; |g(x)| \le K_2, \quad \forall x \in X. \text{..............................}(1)$$

Again, $\{f_n\}$, $\{g_n\}$ converges uniformly on X, then for any $\varepsilon > 0$, there exists natural numbers $m_1(\varepsilon)$ & $m_2(\varepsilon)$ such that

$$\forall x \in X, |f_n(x) - f(x)| < \frac{\varepsilon}{2K_2}, \quad \forall n \ge m_1. \text{..............................}(2)$$ and

$$\forall x \in X, |g_n(x) - g(x)| < \frac{\varepsilon}{2K_1}, \quad \forall n \ge m_2. \text{..............................}(3)$$

Let $m = \max\{m_1, m_2\}$, then for all $x \in X$, we have

$$|f_n g_n(x) - fg(x)| = |f_n(x)\{g_n(x) - g(x)\} + g(x)\{f_n(x) - f(x)\}|$$

$$\le |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$$

$$< K_1 . \frac{\varepsilon}{2K_1} + K_2 . \frac{\varepsilon}{2K_2}, \quad \forall n \ge m. \quad \text{, using } (1), (2) \& (3).$$

$\Rightarrow \{f_n g_n\}$ converges uniformly to fg on X. **(Proved)**

Ex: For each $n \in N$, let $f_n(x) = x - \frac{2}{n}$ and $g_n(x) = x + \frac{3}{n}$, $x \in [0, \infty)$. Show that $\{f_n\}, \{g_n\}$ are uniformly convergent on $[0, \infty)$, but the sequence $\{f_n g_n\}$ is not uniformly convergent on $[0, \infty)$.

Solution: Since $f(x) = \lim_{n \to \infty} f_n(x) = x$ & $g(x) = \lim_{n \to \infty} g_n(x) = x$, $x \in [0, \infty)$.

Let $M_n = \sup\{|f_n(x) - f(x)| : x \in [0, \infty)\} = \frac{2}{n}$. and

$$M_n' = \sup\{|g_n(x) - g(x)| : x \in [0, \infty)\} = \frac{3}{n}.$$

Since $\lim_{n \to \infty} M_n = 0$, $\lim_{n \to \infty} M_n' = 0$.

$\Rightarrow$ both the sequences $\{f_n\}, \{g_n\}$ are uniformly convergent on $[0, \infty)$.

Now, $f_n g_n(x) = \left(x - \frac{2}{n}\right)\left(x + \frac{3}{n}\right) = x^2 + \frac{x}{n} - \frac{6}{n^2}$.

$$\therefore fg = \lim_{n \to \infty} f_n g_n(x) = x^2, \quad x \in [0, \infty).$$

Let $M_n'' = \sup\{|f_n g_n(x) - fg(x)| : x \in [0, \infty)\}$

$= \sup\left\{\left|\frac{x}{n} - \frac{6}{n^2}\right| : x \in [0, \infty)\right\}$ this does not $\to 0$ as $n \to \infty$ for very large x.

Thus the sequence $\{f_n g_n\}$ is not uniformly convergent on $[0,\infty)$. **(Proved)**

Theorem : Let $I=[a, b]$ be a closed bounded interval and for each $n \in N$, $f_n : I \to R$ be R-integrable on I. If the sequence $\{f_n\}$ converges uniformly to a function f on I then f is R-integrable on I and moreover, the sequence

$$\left\{\int_a^b f_n(x)dx\right\} \text{ converges to } \int_a^b f(x)\,dx$$

$$\text{i.e., } \lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b\left\{\lim_{n\to\infty} f_n(x)\right\}dx = \int_a^b f(x)\,dx.$$ **[C.H. '97, '03 V.H.12]**

Proof : Let $\in > 0$ be given.

Since $\{f_n\}$ is uniformly convergent on [a, b] to the function f, then $\exists$ a natural number $k(\in)$ such that

$$\forall x \in [a,b],\ |f_n(x) - f(x)| < \frac{\in}{4(b-a)},\ \forall n \geq k.$$

$$\therefore \forall x \in [a,b],\ |f_k(x) - f(x)| < \frac{\in}{4(b-a)} \qquad \text{---(1)}$$

$$Or,\ f_k(x) - \frac{\in}{4(b-a)} < f(x) < f_k(x) + \frac{\in}{4(b-a)},\ \forall x \in [a,b] \qquad \text{---(2)}$$

Since, f_k is integrable on [a, b].

Then $\exists$ a partition $p = \{a = x_0 < x_1 < .... < x_n = b\}$ of [a, b] such that

$$U(p, f_k) - L(p, f_k) < \frac{\in}{2} \qquad \text{---(3)}$$

Let $M'_r = \sup_{x\in[x_{r-1},x_r]} f_k(x)$, $m'_r = \inf_{x\in[x_{r-1},x_r]} f_k(x)$, for $r = 1,2,3,.......,n$.

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Then, from (2), we have

$$M_r \leq M'_r + \frac{\in}{4(b-a)} \text{ and } m_r \geq m'_r - \frac{\in}{4(b-a)}.$$

$$\therefore M_r - m_r \leq M'_r - m'_r + \frac{\in}{2(b-a)},\ \text{for } r = 1,2,3,.......,n.$$

$$\text{Thus, } \sum_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) \leq \sum_{r=1}^{n}(M'_r - m'_r)(x_r - x_{r-1}) + \frac{\in}{2(b-a)}\sum_{r=1}^{n}(x_r - x_{r-1})$$

Thus the sequence $\{f_n g_n\}$ is not uniformly convergent on $[0,\infty)$. **(Proved)**

Theorem : Let I = [a, b] be a closed bounded interval and for each $n \in N$, $f_n : I \to R$ be R-integrable on I. If the sequence $\{f_n\}$ converges uniformly to a function f on I then f is R-integrable on I and moreover, the sequence

$$\left\{\int_a^b f_n(x)dx\right\} \text{ converges to } \int_a^b f(x)\,dx$$

$$\text{i.e., } \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left\{\lim_{n\to\infty} f_n(x)\right\} dx = \int_a^b f(x)\,dx..$$ [C.H.'97,'03 V.H.12]

Proof : Let $\in > 0$ be given.

Since $\{f_n\}$ is uniformly convergent on [a, b] to the function f, then $\exists$ a natural number $k(\in)$ such that

$$\forall x \in [a,b],\ |f_n(x) - f(x)| < \frac{\in}{4(b-a)},\ \forall n \geq k.$$

$$\therefore \forall x \in [a,b],\ |f_k(x) - f(x)| < \frac{\in}{4(b-a)} \text{ ------------------------(1)}$$

$$Or,\ f_k(x) - \frac{\in}{4(b-a)} < f(x) < f_k(x) + \frac{\in}{4(b-a)},\ \forall x \in [a,b] \text{ ------------(2)}$$

Since, f_k is integrable on [a, b].

Then $\exists$ a partition $p = \{a = x_0 < x_1 < .... < x_n = b\}$ of [a, b] such that

$$U(p, f_k) - L(p, f_k) < \frac{\in}{2} \text{ ------------------------ (3)}$$

Let $M'_r = \sup_{x\in[x_{r-1},x_r]} f_k(x)$, $m'_r = \inf_{x\in[x_{r-1},x_r]} f_k(x)$, for $r = 1, 2, 3, ..., n$.

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Then, from (2), we have

$$M_r \leq M'_r + \frac{\in}{4(b-a)} \text{ and } m_r \geq m'_r - \frac{\in}{4(b-a)}.$$

$$\therefore M_r - m_r \leq M'_r - m'_r + \frac{\in}{2(b-a)},\ \text{for } r = 1, 2, 3, ..., n.$$

$$\text{Thus, } \sum_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) \leq \sum_{r=1}^{n}(M'_r - m'_r)(x_r - x_{r-1}) + \frac{\in}{2(b-a)}\sum_{r=1}^{n}(x_r - x_{r-1})$$

Or, $U(p, f) - L(p, f) \le U(p, f_k) - L(p, f_k) + \frac{\epsilon}{2}$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}, \quad \text{by (3)} = \epsilon.$$

This proves that f is R-integrable on [a, b].

Second part : Let us choose $\varepsilon > 0$.

Since, $\{f_n\}$ is uniformly converges to f on [a, b]. Then $\exists$ a natural number k such that

$\forall x \in [a, b]$; $|f_n(x) - f(x)| < \frac{\varepsilon}{(b-a)}$, $\forall n \ge k$. --(4)

Now, $\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| = \left|\int_a^b \{f_n(x) - f(x)\}dx\right| \le \int_a^b |f_n(x) - f(x)|dx < \int_a^b \frac{\varepsilon}{b-a} dx = \varepsilon$.

, using (4).

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Note1 : For each $x \in [a, b]$, $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

Note2 : If $\{f_n\}$ is sequence of integrable function converging to f on [a, b] and if

$\int_a^b f(x)dx \ne \lim_{n\to\infty} \int_a^b f_n(x)\, dx$. , then $\{f_n\}$ can not converges uniformly to f.

Note3 : The above statement of the theorem is sufficient condition but not a necessary condition.

i.e., if $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

Then the sequence $\{f_n\}$ may not be uniformly convergent on [a, b].

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in [0, 1]$.

This sequence $\{f_n\}$ converges point wise on [0, 1] to the function f, where

$$f(x) = 0, \quad 0 \le x \le 1.$$

Each f_n is integrable on [0, 1] and also f is integrable on [0, 1].

Now, $\int_0^1 f_n(x)dx = \int_0^1 \frac{nx}{1+n^2x^2} dx = \left[\frac{1}{2n}\log(1+n^2x^2)\right]_0^1 = \frac{1}{2n}\log(1+n^2)$.

$$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{\log(1+n^2)}{2n} \quad \left(\frac{\infty}{\infty} \textit{ form}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+n^2}.2n}{2} = \lim_{n\to\infty} \frac{n}{1+n^2} \quad \left(\frac{\infty}{\infty} \textit{ form}\right), \quad \text{using L'Hospital rule.}$$

$$= \lim_{n\to\infty} \frac{1}{2n} = 0.$$

And, $\int_0^1 f(x)dx = 0.$ Thus, $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

But the sequence $\{f_n\}$ is not uniformly convergent to f on [0, 1].

Ex: Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$ is not uniformly convergent on [0,1]

Solution : Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{nx}{1+\frac{nx^2}{1!}+.....} = 0, \ \forall x \in [0,1].$

Thus, $\int_0^1 f(x)dx = \int_0^1 0dx = 0$ ---(1)

Also, $\int_0^1 f_n(x)dx = \int_0^1 nxe^{-nx^2}dx$

Put $nx^2 = t \ \therefore 2nxdx = dt$.

$$\therefore \int_0^1 f_n(x)dx = \frac{1}{2}\int_0^n e^{-t}dt = \frac{1-e^{-n}}{2}.$$

$$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \left(\frac{1-e^{-n}}{2}\right) = \frac{1}{2} \quad \text{------------(2)}$$

2nd Process :

$e^{-nx^2} < \frac{n^2x^4}{2}$

$\Rightarrow nxe^{-nx^2} < \frac{2}{nx^3}$

Let $k = \left[\frac{2}{nx^2}\right] + 1$

as $x \to 0$, $k \to \infty$

From (1) and (2), we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2} \neq 0 = \int_0^1 f(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

Which shows that the given sequence is not uniformly convergent in [0, 1].

Ex: Let $f_n(x) = n^2x(1-x^2)^n$, $x \in [0,1]$. Show that the sequence $\{f_n\}_n$ converges pointwise to f on $[0,1]$, where $f(x) = 0$, $\forall x \in [0,1]$. But the sequence is not uniformly convergent.

Solution: Since $f_n(0) = 0 = f_n(1)$, $\forall n \in N$.

$$\therefore \lim_{n\to\infty} f_n(x) = 0 \quad for \ x = 0 \text{ and } x = 1.$$

Now, for $0<x<1$, $0<x^2<1 \Rightarrow 0<1-x^2<1$.

Let $1-x^2=\dfrac{1}{1+a}$, $a>0$, then $f_n(x)=\dfrac{n^2x}{(1+a)^n}$.

Since $(1+a)^n > \dfrac{n(n-1)(n-2)}{3!}a^3$. $\Rightarrow 0<f_n(x)<\dfrac{\frac{6x}{n}}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)a^3}$, $0<x<1$.

By Sandwich theorem, we have

$$\lim_{n\to\infty} f_n(x)=0,\ \ 0<x<1.$$

Thus the sequence $\{f_n\}_n$ converges point wise to f on $[0,1]$

, where $f(x)=0,\ \forall x\in[0,1]$.

Now, $\int_0^1 f_n(x)\,dx = n^2\int_0^1 x(1-x^2)^n\,dx = \dfrac{n^2}{2}\left[-\dfrac{(1-x^2)^{n+1}}{n+1}\right]_0^1 = \dfrac{n^2}{2(n+1)}$.

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\infty \ \text{ and } \int_0^1 f(x)\,dx=0.$$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

$\Rightarrow \{f_n\}_n$ is not uniformly convergent on $[0,1]$. **(Proved)**

Ex: Let $f:R\to R$ be uniformly continuous on R. For each natural number n,

$$f_n(x)=f\left(x+\frac{1}{n}\right),\ \ x\in R.$$

Prove that the sequence $\{f_n\}$ is uniformly convergent on R.

Solution : For all $x\in R$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f\left(x+\frac{1}{n}\right) = f(x)$, Since f is continuous at x.

$\therefore$ The sequence $\{f_n\}$ converges to a function f on R.

Let $\epsilon>0$. Since f is uniformly continuous on R, then $\exists$ a positive δ such that $\forall x,u\in R$,

There exists a natural number k such that $0<\frac{1}{n}<\delta,\ \forall n\geq k$.

$$|x-u|<\delta \Rightarrow |f(x)-f(u)|<\epsilon \ \text{---(1)}$$

It follows from (1) that for all $x\in R$,

$$\left|f\left(x+\frac{1}{n}\right)-f(x)\right|<\in, \quad \forall n \geq k$$

i.e., $\forall x \in R$, $|f_n(x)-f(x)|<\in, \quad \forall n \geq k$

This proves that the sequence $\{f_n\}$ is uniformly convergent on R to f.

Ex: For each $n \geq 2$, let $f_n(x) = n^2x, \; 0 \leq x \leq \frac{1}{n}$.

$$= -n^2x + 2n, \; \frac{1}{n} < x < \frac{2}{n}$$

$$= 0, \; \frac{2}{n} \leq x \leq 1.$$

(*i*) Show that the sequence $\{f_n\}_{n=2}^{\infty}$ converges to a function f on [0, 1].

(*ii*) Show that the converges is not uniform on [0, 1] by establishing that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx.$$ **[V.H.'01, C.H.'05]**

Solution : (*i*) Let $c \in (0, 1)$ be any point.

Then $\exists$ a natural number $p > 2$ such that $0 < \frac{2}{p} < c$ and therefore $\frac{2}{n} < c < 1, \; \forall n \geq p$..

Thus, $f_p(c) = 0$ and therefore $f_n(c) = 0, \; \forall n \geq p$.

$\therefore \lim_{n\to\infty} f_n(x) = 0, \; \forall x \in (0,1)$.

Also, when $x = 0$ and $x = 1$, the sequence is $\{0, 0, \ldots\ldots\}$ which converges to 0.

Thus the sequence $\{f_n\}_2^{\infty}$ converges to f on [0, 1]

, where $f(x) = 0, \; \forall x \in [0,1]$.

(*ii*) Now, $\int_0^1 f_n(x)\,dx = \int_0^{\frac{1}{n}} n^2x\,dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (-n^2x+2n)dx + \int_{\frac{2}{n}}^{1} 0\,dx. = \left(\frac{n^2x^2}{2}\right)_0^{\frac{1}{n}} + \left[-\frac{n^2x^2}{2}+2nx\right]_{\frac{1}{n}}^{\frac{2}{n}} + 0$

$$= \frac{1}{2} - 2 + 4 + \frac{1}{2} - 2 = 1, \; \forall n \geq 2$$

$\because \lim_{n\to\infty}\int_0^1 f_n(x)dx = 1.$ and $\int_0^1 f(x)dx = \int_0^1 0\,dx = 0.$

Thus, $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$.

It follows that the sequence $\{f_n\}_2^\infty$ is not uniformly convergent on [0, 1].

Ex: If $\{f_n\}$ converges uniformly to f on S ($\subseteq$R) and the function g is bounded on S. Show that $\{f_n g\}$ converges uniformly to fg on S.

Solution : Since, the function g is bounded on S, then $\exists$ a positive real number B such that

$|g(x)| \le B, \ \forall x \in S$ -- (1)

Let us choose $\in > 0$.

Since, the function $\{f_n\}$ converges uniformly to f on S, then $\exists$ a natural number k such that

$\forall x \in S, |f_n(x) - f(x)| < \frac{\in}{B}, \ \forall n \ge k$ -- (2)

Now, $|f_n g(x) - fg(x)| = |f_n(x)g(x) - f(x)g(x)| = |g(x)||f_n(x) - f(x)|$ ....(3)

From (3), using (1) and (2) we have

$\forall x \in S, \ |f_n g(x) - fg(x)| < B.\frac{\in}{B}, \ \forall n \ge k = \in.$

This proves that the sequence $\{f_n g\}$ converges uniformly to fg on S.

Ex: Let ϕ be continuous on [0, 1] and $f_n(x) = x^n \phi(x), \ 0 \le x \le 1.$ C.H.' 07,05, 02, 03 V.H'08

Prove that $\{f_n\}$ converges uniformly on [0, 1] if and only if $\phi(1) = 0$.

Solution : Since ϕ is continuous on [0, 1], therefore it is bounded on [0, 1].

Since, $\lim_{n\to\infty} x^n = 0$ for $0 \le x < 1$ and ϕ is bounded on [0, 1].

$\therefore$ We have, $\lim_{n\to\infty} x^n \phi(x) = \lim_{n\to\infty} f_n(x) = 0, \ 0 \le x < 1.$

For x = 1, the sequence is $\{\phi(1), \phi(1), ...\}$ and this converges to $\phi(1)$.

Thus, the sequence $\{f_n\}$ converges to the function f on [0, 1], where

$$f(x) = 0, \text{ for } 0 \le x < 1$$
$$= \phi(1), \text{ for } x = 1.$$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|, \ \forall n \in N.$

If $\phi(1) = 0$, then $\exists$ a real number $k \in (0,1)$ such that $M_n = k^n |\phi(k)|, \ \forall n \in N.$

$\therefore \lim_{n\to\infty} M_n = 0$, since $0 < k < 1$ and $\phi(k)$ is finite.

Thus if $\phi(1) = 0$, then the sequence $\{f_n\}$ is uniformly convergent on [0, 1].

Conversely, let the sequence $\{f_n\}$ be uniformly convergent on [0,1].

Since, ϕ is continuous on $[0, 1]$, each f_n is continuous on $[0, 1]$ and the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

$\therefore$ f is continuous on $[0, 1]$.

Since, $\lim_{x\to 1-} f(x) = 0$ and f is continuous at $x = 1$, it follows that $\phi(1) = 0$.

Ex. Let D be a finite subset of R. If a sequence of real valued function $\{f_n\}$ converges point wise on D. Shows that the sequence $\{f_n\}$ is uniformly convergent on D. **C.H. 03**

Solution : Let $D = \{x_1, x_2, ..., x_n\}$ be finite.

Since, the sequence $\{f_n\}$ converges point-wise on D, let the limit function be f.

Let us choose $\in > 0$.

Then, $\exists$ natural numbers $k_1, k_2, ..., k_r$

Such that,

$|f_n(x_i) - f(x_i)| < \in, \forall n \geq k_i$ and $i = 1, 2, 3, ..., r$.

Let $k = \max\{k_1, k_2, ..., k_r\}$

Then we have,

$\forall x \in D, \ |f_n(x) - f(x)| < \in, \ \forall n \geq k.$

This imply that the sequence $\{f_n\}$ is uniformly convergent on D.

Ex. Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx}$, $x \geq 0$ converges uniformly to a function f. Examine whether $\lim_{n\to\infty} f'_n(0) = f'(0)$. **[V.H.2k]**

Solution : For $x = 0$, the sequence is $\{0, 0, ...\}$ and this converges to 0.

For $x > 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n}+x} = 0.$

Thus the limit function is $f(x) = 0, \ \forall x \geq 0$

Now, $\forall x \geq 0 \ |f_n(x) - f(x)| = \frac{x}{1+nx} < \frac{1}{n}$ as $x \geq 0$.

Let us choose $\in > 0$.

Then, $|f_n(x) - f(x)| < \in, \ \forall n > \frac{1}{\in}$

Let $k = \left[\frac{1}{\in}\right] + 1$. Then k is a natural number and $\forall x \geq 0$, $|f_n(x) - f(x)| < \in, \ \forall n \geq k$.