Theorem 1:—

Every non empty subset of $\mathbb{N}$ has a least element.

Proof:— Let S be a non empty subset of $\mathbb{N}$

Let, $k \in S$

Let us consider, $T = \{x \in S : x \leq k\}$

$\therefore T \subseteq \{1, 2, \ldots k\} = S$

$\therefore T$ is a finite set.

$\therefore T$ has a least element say (m).

$\therefore m \leq k$

Let, $s \in S$

Then s either, $s > k$ or $s \leq k$.

Case I if $s > k$ then,

$\therefore m \leq k < s$

$\Rightarrow m < s$

Case II $s \leq k$

$\Rightarrow s \in T$

Since, m is the least element of T.

$m \leq s \vee s \in T$

$\Rightarrow m \leq s \vee s \in S$

Hence, S has a least element m. (Proved)

Theorem 2:—

Let S be the subset of $\mathbb{N}$ then,

(i) $1 \in S$

(ii) $k \in \mathbb{N}$, such that $k \in S \Rightarrow k+1 \in S$

then, $S = \mathbb{N}$.

Proof:— Let, S be a subset of $\mathbb{N}$. Therefore, consider, T be the set of all elements which are not in S.

Therefore, We assume that, T be a non empty. Then by well ordering property, we get, a least element m (say)

$\therefore m \in T \Rightarrow m \notin S$ also, Since $1 \in S \Rightarrow m > 1$. Therefore, $m-1 \in \mathbb{N}$

$m-1 \notin T$ then, $m-1 \in S$

by (ii) $(m-1)+1 \in S \Rightarrow m \in S$ [which is a contradiction] because, $m \in T$. So, our assumption is wrong

1 is a empty subset.
Hence, we observe that, $1 \in S$
(ii) $k \in S \Rightarrow (k+1) \in S$. (Proved)

(*) Theorem 3 :=

(*) Linear Diophatine Eq" :—

$$ax + by = c$$

(*) Theorem :—
$ax+by = c$ admits a sol" iff $d \mid c$ where, $d = \gcd(a, b)$

3.
(*) Theorem :—
Let, a, b both are not zero. and $\gcd(a, b) = d$ the
$\exists$ integers $u, v \in \mathbb{Z}$ such that, $d = au + bv$

Proof :— Let, a, b not both are zero.
Let consider, $S = \{ ax+by : x, y \in \mathbb{Z};\ ax+by > 0 \}$

$\therefore S \subseteq \mathbb{N}$
by well ordering property, S be a non empty subset of $\mathbb{N}$.
because, without loss of generality, we get

$|a| = |a| = ax+by$, if $x=1, y=0$ & $a>0$,
$-a = -ax+by$, $x=-1, y=0, a<0$,

$\therefore S \neq \phi$
Therefore, S has a least element (say d)

$\therefore d \in S$
Thus, $\Rightarrow d = au+bv$ for $u, v \in \mathbb{Z}$.
Conversely, Let, $d = au+bv$ $(u, v \in \mathbb{Z})$ by division algorithm

$a = dq + r$, $0 \le r < d$, $q \in \mathbb{Z}$.

If possible let, $0 < r < d$

$\therefore a = dq + r$

$\Rightarrow a - dq = r$

$\Rightarrow a - q(au + bv) = r$

$\Rightarrow a - aqu - qbv = r$

$\Rightarrow a(1 - qu) - qbv = r$

$\Rightarrow a(1-qu) + (-qv)b = r$, where, $u = 1 - qu$, $v = -qv$

$\Rightarrow r \in S$, with $0 < r < d$

= which is a contradiction, because, d is the least element of S

Then, $r = 0$.

$\therefore a = dq$

$\Rightarrow d \mid a$

Similarly, we can prove that, $d \mid b$

Hence, $d \mid a$, $d \mid b$...

d then $d = \gcd(a, b)$...

Hence, $\gcd(a,b) = d = au + bv$

$\Rightarrow d = au + bv$, $u, v \in \mathbb{Z}$.

Theorem:—

The number of prime number is Infinite.

Proof:— We assume that,

Let, $p_1, p_2, \ldots p_k$ be the finite prime numbers.

Let, n be a positive integer $n = p_1 p_2 \ldots p_k + 1$

clearly see that, $n > 1$

By fundamental theorem of arthmetic $\exists$ a prime divisor p such that, $p \mid n$.

also, $p \mid p_1 p_2 \ldots p_k \; \forall k \in \mathbb{N}$.

Therefore, $p \mid n - p_1 p_2 \ldots p_k$

$\Rightarrow p \mid 1$ [which is a contradiction]

$\Rightarrow p = 1$

Hence, the number of prime number is infinite.

5. Theorem-5:- (Fundamental theorem of arithmetic):-

Every positive integer is either 1 or prime or product of prime

Proof:-

Let, n be a positive integer.

Case-1

n = 1 then the theorem is over.

Case-2

Let, n = 2 then, P(2) is prime.

Therefore, P(2) P(3) ... P(k) is true. (assume that)

Let, n = k+1

If n = k+1 is prime, then, P(k+1) is true.

If n = k+1 is a composite number then,

then, n = k+1 = r.s , where, $1 < r < k+1$ & $1 < s < k+1$

i.e, $2 \le r \le k$ & $2 \le s \le k$.

Therefore, P(n) = P(r).P(s) is always a product of prime number. P(n) = P(r).P(s) is true for all n.

Hence, by mathematical induction, we get, P(n) is true for all n. $\forall n \in \mathbb{Z}$.

6. Archemedean property:-

Let, $x, y \in \mathbb{R}$ and $x > 0$, $y > 0$ then there exist a natural number n such that, $ny > x$

Proof:- Let, $x, y \in \mathbb{R}$, If possible let, $\exists$ no natural number $n \in \mathbb{N}$ such that, $ny \le x$, where, x is an upper bound of

$S = \{ y \in \mathbb{R} : ny \le x \}$, x is an upper bound of S.

$\therefore$ S is an bounded above set, then, Sup S are exist

$\therefore$ Sup S = b. $\Rightarrow ny \le b$

$\therefore y > 0$

$-y < 0$

$\Rightarrow b - y < 0$

$\Rightarrow b - y < ny \le b$ then $b \cdot y - ny < 0 \Rightarrow y(1+n) > b$

$\Rightarrow b - y(1+n) < 0$