A complete bipartite graph is denoted by $K_{m,n}$ where m is the number of vertices in V_1 & n be the number of vertices in V_2

Notes: A complete Bipartite graph K_{mn} has $m+n$ vertices and mn edges.

Q Show that the maximum number of edges in a complete bipartite graph with n vertices is $\frac{n^2}{4}$.

⇒ Let G be a complete Bipartite graph with n vertices.

Let n_1 & n_2 be the number of vertices in the partition V_1 & V_2 of the vertex set of G.

Since G is complete bipartite graph, so each vertex in V_1 is joined to each vertex in V_2 by exactly one edge.

Then G has $n_1 n_2$ edges, where $(n_1+n_2) = n$.

Since n_1, n_2 be the number of vertices of V_1 & V_2, then n_1 & n_2 must be positive integer.

Let us consider two positive number n_1, n_2 and applying A.M $\geq$ G.M we have

$$\frac{n_1+n_2}{2} \geq (n_1 n_2)^{1/2}$$ the equality occur when $n_1 = n_2$

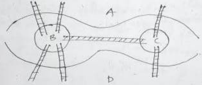
$$n_1 n_2 \leq \left(\frac{n_1+n_2}{2}\right)^2 = \frac{n^2}{4} \quad (\because n_1+n_2 = n)$$

$$\therefore n_1 n_2 \leq \frac{n^2}{4}$$

This implies that the maximum value of $n_1 n_2$ is $\frac{n^2}{4}$ and the maximum value attained when $n_1 = n_2$.

Hence the maximum of edges is $\frac{n^2}{4}$.

- **Euler Graph:**



Konigsberg 7-bridge problem

The problem is to start at any of the 4 land area A, B, C, D

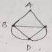
Walk over is each of 7 bridges exactly one and return to the starting point (with out swimming)

⇒ **Defn:**

A closed walk in a graph which contains all the edges of the graph is called an Euler line sometime it is called Euler circuit.

A graph which contain an Euler line is called an Euler graph.

As for example:



- **Theorem:**

A connected graph G is an Euler graph iff the degree of every vertex in G is even.

⇒ Let us consider the given graph G be an Euler graph.

Then G contain an Euler line starting from a vertex v_1 (say) in G.

Let this Euler line travels all the edges of G be $v_1 e_1 v_2 e_2 v_3 e_3 \ldots v_n e_n v_1$

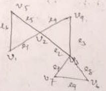
where $v_{n+1} = v_1$. In this Euler line the vertices $v_1, v_2, \ldots v_n$ may not all be distinct and some of this vertices may be repeated but all the edges are distinct.

In this Euler line a pair of successive edges e_i & e_{i+1}, $i = 1, 2, \ldots n-1$, contributes two to the degree of the vertex v_{i+1}

$\therefore$ The vertices $v_2, v_3, \ldots v_n$ are of even degree. Also the vertex v_1 gets a contribution of two to the its degree from the initial & final edges e_1 & e_n. Thus all the vertices are of even degree.

Conversely we assume that all the vertices of the graph G are of even degree

[illegible]



$H: v_1 e_1 v_2 e_5 v_5 e_6 v_1$

$H' = G - H$

$= v_2 e_4 v_4 e_3 v_3 e_2 v_2$

Let v be any vertex in G. Now we construct a work starting at v & going through the edges of G s.t no edges repeated. Since every vertex is of even degree so we can exit from every vertex we enter. Since v is also even degree so starting the work at the vertex v and terminate at also v. If this closed walk H(say) contain all the edges of G then G is an Euler graph.

Again if this closed walk does not contain all the edges of G then we remove from G all the edges in H & obtain a subgraph H' = G-H, of G formed by the remaining edges.

Since both G & H have all their vertices of even degree so the degree of vertices of H' are also even.

Since G is connected, so H' must touch edge at least at one vertex u say. Starting from u we again construct a new graph walk in the graph H'. Since all the vertices of H' are of even degree, So this walk in H' must terminate at the vertex u. Now this walk in H' can be combined with H to form a new walk which starts and term at v. This argument can be repeated untill we obtain an Euler line in G.

Hence the graph G is an Euler graph.

Which of the following graphs are Eulerian.

i) The graph K_5

ii) " " $K_{3,2}$

iii) The petersen graph.

K_5

We know that the degree of every vertices of a completegraph with n vertex is n-1.

So for K_5 the degree of every vertices is $5-1 = 4$, which is even.

Also, K_5 is a connected graph.

Hence the graph K_5 is an Euler graph.

$K_{3,2}$: