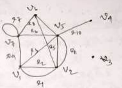
## Graph theory

**Defn of Graph:**

A graph $G$ is a pair $(V, E)$, where $V = \{v_1, v_2, \dots\}$ is a non-empty set whose elements are called vertices or nodes, and $E = \{e_1, e_2, \dots\}$ is a set s.t each element $e_k$ of $E$ is identified with an unordered pair $(v_i, v_j)$ of vertices. The element of $E$ are called edges of $G$.

As for example:




Here $G = (V, E)$, $V = \{v_1, v_2, \dots v_7\}$

$E = \{e_1, e_2, \dots e_{12}\}$

**Parallal edge:**

More than one edge Betn pair of vertices, such edges are called parallal edges. As for example in above graph $e_5, e_{11}$ are parallal edges $e_1, e_2$ are also parallal edges.

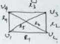
- **Self-loop or loop:**

An edge having the same vertex as both its end vertices is called a self loop or loop. In above graph $e_7$ is a self loop.

- **Simple graph:**

A graph G is said to be a simple graph if it has neither self loop nor a parallal edges.

As for example:



- **Incident and adjacent:**

I Let $e_k$ be an edge joining two vertices $v_i$ & $v_j$ of a graph G then the edge $e_k$ is said to be incident on each ~~edge~~ of its end vertices $v_i$ & $v_j$.

As for example:

In above graph $e_6$ is incident with $v_5$ & $v_7$.

- Two vertices in a graph are said to be adjacent if there exists an edge joining the vertices.

As for example: in above graph $v_1$ & $v_5$ are adjacent but $v_1$, $v_6$ are not adjacent.

● Degree of a vertex:

The degree of a vertex $v$ in a graph G is denoted by $d(v)$ / $deg(v)$ and is equal to the number of edges which are incident on $v$. with self loop counted twice.

In above graph -

$d(v_1) = 4$  $d(v_2) = 5$  $d(v_3) = 0$ (isolated)  $d(v_4) = 1$ → pendent

$d(v_5) = 6$  $d(v_6) = 3$  $d(v_7) = 5$  ~~$d(v_8)$~~

Pendent vertex:

A vertex is said to be pendent vertex if degree is 1.

In above graph $v_4$ is a pendent vertex

◎ Isolated vertex:

A vertex having no edge incide on eachit is called an isolated vertex. i.e a vertex v is said to an isolated vertex if degree is zero.

As for example:

In above graph $v_3$ is an isolated vertex.

◎ Null graph:

A graph $G = (V, E)$ is said to be a null graph if the set of vertices V is non empty but the set of edges E are empty. i.e a null graph. Every vertex is an isolated vertex.

As for example:


◎

Theorem:

The sum of degree of all vertices in a graph G is equal to the twice the number of edges.

Proof:

Let G be a graph with e edges and n vertices.

Also $v_1, v_2, \ldots, v_n$ be the n vertices of G.

Since each edge each incident on two vertices, it contribute 2 to the sum of the degrees of the graph.

Thus the sum of degrees of all the vertices in G is twice the number of edges in G. ie $\sum_{i=1}^{n} d(v_i) = 2e$.

This result is also known as hand shaking lemma.

- Theorem:

The number of odd vertices, a vertices of odd degree in a graph is always even

Let G be a graph and e is the total number of edge and n is the total number of vertices say $v_1, v_2 \ldots v_n$

We know that the sum of all degree of all the vertices in a graph G is is equal to the twice the total number of edges ie $\sum_{i=1}^{n} d(v_i) = 2e$

We can assume that the degree first r vertices $(r \le n)$ $v_1, v_2 \ldots$ ①
be even degree and those of the remaining $(n-r)$ vertices be odd degree.
Then from ① we have,

$$\sum_{i=1}^{r} d(v_i) + \sum_{i=r+1}^{n} d(v_i) = 2e \qquad ②$$

The R.H.S of two is always even. Also the first summation on LHS is also even because each term in this sum is even.
Hence the 2nd sum on LHS must be even. i.e ~~Summation~~

i.e $\sum_{i=r+1}^{n} d(v_i) =$ an even number ③

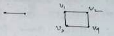
Since each term $d(v_i)$ in ③ is odd, the total number of terms in the summation must be even. (To make the sum an even number)
Hence $(n-r)$ is the even number.
Therefore the number of odd vertices is even.

Hence proof.

## Regular graph:

A graph G is said to be a regular graph if all the vertices are off equal degree

As for example:




Show that the maximum degree of any vertex in a simple graph with n vertices is (n-1)

⇒ Since a simple graph has no self loop & no parallal edges. So the maximum case in a simple graph one vertex can be adjacent with the remaining vertices.

In a simple graph with n vertices, any one vertex can be adjacent remaining (n-1) vertices. So maximum degree of this vertex is n-1, therefore in a simple graph, the maximum degree of any vertex is n-1.

Show that the maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

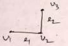
⇒ A simple graph has maximum number of edges only when there is an edge bet every pair of vertices. Out of n vertices any two vertices can be joined in $^nC_2$ ways i.e in $\frac{n(n-1)}{2}$ ways.

Therefore the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

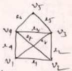
## ⊘ Subgraph:

Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is said to be a subgraph of G if $V' \subseteq V$ & $E' \subseteq E$ s.t an edge $(v_i, v_j)$ is in $E'$ only if $v_i$ & $v_j$ are present in $V'$

⇒




$H = (V', E')$
$V' = \{v_1, v_2, v_3\}$
$E' = \{e_1, e_2\}$



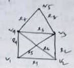

$V = \{v_1, v_2, \ldots v_5\}$
$E = \{e_1, e_2, \ldots e_8\}$

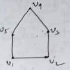
$\therefore V' \subseteq V$ & $E' \subseteq E$

## Spanning Subgraph:

A Subgraph H of a graph G is said to be a spanning subgraph if all the vertices of G are present in the subgraph H.

Example:





$G = (V, E)$
$V = \{v_1, v_2 \dots v_5\}$
$E = \{e_1, e_2 \dots e_8\}$.

$H = (V, E')$

## Complement of a Subgraph:

Let $H = (V', E')$ be a subgraph of a graph $G = (V, E)$, then the complement of the subgraph H w.r.t the graph G is the subgraph of G say $\bar{H}$, where $\bar{H} = (V, E - E')$

As for example:

(a)

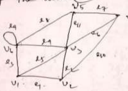
• Walk, path & circuit :

Walk :

A walk in a graph $G=(V,E)$ is a finite alternative seq^n of vertices and edges, $v_0 e_1 v_1 e_2 v_2 e_3 v_3 \ldots v_{k-1} e_k v_k$, begining and ending with vertices s.t. $v_{i-1}$ and $v_i$ are the end vertices of an edge $e_i$, $i=1,2,\ldots,k$ and all the edges are distinct.

The vertices $v_0$ & $v_k$ with which a walk begening and ending are called its end or terminal vertices. All other vertices ~~At other~~ are called internal vertices of the walk.

A.F.E :




i) $v_1 e_1 v_2 e_2 v_3 e_{11} v_5$

ii) $v_1 e_5 v_3 e_6 v_4 e_7 v_5 e_8 v_6 e_3 v_1$

**Open walk:**

A walk is said to be an open walk if its end or terminal vertices are distinct. In above example, ①, ② are open walk.

**Closed walk:**

A walk is said to be an closed walk if it is not open. Then a closed walk a walk which begining and ending at the same vertex.

In above example ② are closed walk.

**Path:**

An open walk in which no vertex apper more than one is called a path. ie in a path all the verties are distinct.

In above example ① is path But ② (tammy khata) is open walk but not path.

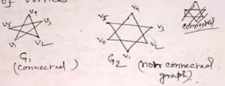
**Length of a path:**

The number of edges in a path is called the length of the path.

◉ Circuit:
A closed walk is said to be circuit if all the internal vertices are distinct.

In above example ④ (taking walk) is circuit but ③ is not.

◉ Connected graph:
A graph G is said to be connected if ∃ a path betn every pair of vertices




◉ Disconnected graph
A graph which is not a connected graph is called a disconnected graph.
Thus a graph G is disconnected if we can find a pair of vertices $v_i$ & $v_j$ in G s.t there is no path betn $v_i$ & $v_j$