Class notes by Amalansu Sekhar Pattanayak. page1

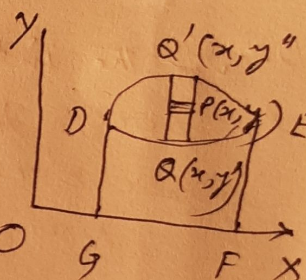
• Green's Theorem and ~~Stokes~~ Stoke's theorem; Divergence Theorem: 4th - Sem (H) Paper C9T. and Sums.

Green's Theorem in a Plane.

If S be a closed region in the xy-plane bounded by a simple closed curve C and if M and N be functions of x and y which are continuous, having continuous derivative in S, Then

$$\oint_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

where C is traversed in counter clock-wise sense.



Let C be a closed curve enclosings such That any st. line parallel to the co-ordinate axes cuts it in atmost two Points D and E ~~the~~ on the curve where the tangents are parallel to the y-axis DG and DF are ordinates of these points and $OG = a$, $OF = b$. Let $P(x,y)$ be a general point in S. we introduce an element of area at P lying in a strip parallel to the y-axis. The ends of strip cut C at Q and Q' whose co-ordinates are (x, y') and (x, y'') respectively. The points D and E divide the curve C in two parts DQE and $DQ'E$. Then,

$$\iint \frac{\partial M}{\partial y}\,dx\,dy = \int_a^b \int_{y'}^{y''} \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b \left[M(x,y)\right]_{y'}^{y''} dx$$

$$= \int_a^b M(x,y'')\,dx - \int_a^b M(x,y')\,dx = -\int_b^a M(x,y'')\,dx - \int_a^b M(x,y')\,dx$$

$$= -\int_a^b M(x,y')\,dx - \int_b^a M(x,y'')\,dx = -\oint_C M\,dx$$

where the integration are taken over the arcs DQE and $EQ'D$

Next

Thus $\oint_C M dx = -\iint_S \frac{\partial M}{\partial y} dx\,dy$ - - - - ①

Similarly by drawing strips parallel to x-axis and proceeding as before, we can show that

$\oint_C N dy = \iint_S \frac{\partial N}{\partial x} dx\,dy$ - - - ②

adding ① and ② we get

$$\oint_C (M dx + N dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Proved.

N.B. ① If the integral around any closed ~~curve~~ path be zero, the $\nabla \times F = 0$ so that the field is Conservative.

Now $\nabla \times F = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

② Green's Theorem in a plane is a special case of Stoke's Theorem.

: Green's Theorem in plane in vector notation.

We have $M dx + N dy = (M\hat{i} + N\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j}) = F \cdot dr$ where $F = M\hat{i} + N\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j}$

Now $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial x}\hat{i} + \frac{\partial M}{\partial z}\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$

Hence $(\nabla \times F) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$. Hence in vector notation, Green's theorem in plane becomes

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{k}\, dS = \iint_S (\nabla \times F) \cdot \hat{k}\, dx\,dy$$

Stoke's Theorem

Stoke's theorem: If F be a Continuously differentiable Vector Point function and S be the Surface bounded by the closed Curve C, then Mathematically

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \oint_C \vec{F}\cdot d\vec{r} \quad \text{or.}$$

$$\iint_S (\nabla\times F)\cdot\hat{n}\,ds = \oint_C F\cdot dr$$ where C is traversed in +ve direction and $\hat{n}$ is the unit vector outward normal drawn to S.

ie Surface integral of the Component of curl $\vec{F}$ along the normal to the surface S, taken ~~the~~ over the surface bounded by curve C is equal to the line integral of the vector Point function $\vec{F}$ ~~along~~ taken along the closed curve C.

Volume Integral: Let $\vec{F}$ be a vector Point function and volume V enclosed by closed surface. Then Volume integral $V = \iiint_V \vec{F}\,dv = \iiint \vec{F}\,dxdydz$.

Gauss' divergence Theorem or Divergence Theorem:— The volume integral of the ~~diverse~~ divergence of a Vector $\vec{F}$ in a vector field, taken through a given volume is equal to the surface integral of the normal Component of F over surface enclosing the volume

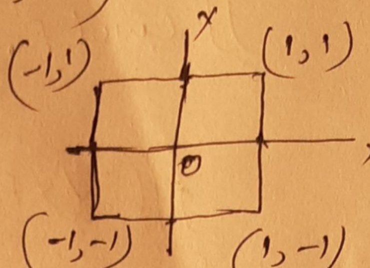
that is $$\iiint_V \text{div}\,\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

(Relation between volume integral and surface integral)

For Proof Consult Vector Analysis — by Ghosh & Maity and Chakravorty & Ghosh

Sums

EX I use Stokes' theorem to determine $\oint(xy$ $\oint(xy\,dx+xy^2dy)$. where c is a square having vertices $(1,1),(-1,1),(-1,-1),(1,-1)$ in the xy-plane



Sol^n: Given $\oint_c (xy\,dx+xy^2dy)$

$= \oint_c (xy\hat{i}+xy^2\hat{j})\cdot(dx\hat{i}+dy\hat{j})$

$= \oint_c \vec{F}\cdot d\vec{r}$ where $\vec{F}=xy\hat{i}+xy^2\hat{j}$ and $d\vec{r}=dx\hat{i}+dy\hat{j}$.

$\because \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = 0\hat{i}+0\hat{j}+\cancel{\hat{k}\,\partial y^2}$
$+\hat{k}\{\frac{\partial}{\partial x}(xy^2-\frac{\partial}{\partial y}(xy)\}$
$=\hat{k}(y^2-x)$

By Stokes' Therorem we know

$\oint_c \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds = \iint_S \hat{k}(y^2-x)\cdot\hat{k}\,dx\,dy$

$=\iint_S (y^2-x)\,dx\,dy$ where x varies from -1 to 1
y " from -1 to 1

Hence $\oint_c \vec{F}\cdot d\vec{r} = \int_{-1}^{1}\int_{-1}^{1}(y^2-x)\,dx\,dy = \int_{-1}^{1}\left[\int_{-1}^{1}(y^2-x)\,dx\right]dy$

$=\int_{-1}^{1}\left[y^2x-\frac{x^2}{2}\right]_{-1}^{1}dy = \int_{-1}^{1}\left[(y^2-\frac{1}{2})-(-y^2-\frac{1}{2})\right]dy$

$=\int_{-1}^{1}(y^2-\frac{1}{2}+y^2+\frac{1}{2})\,dy = \int_{-1}^{1}2y^2dy = 2\left[\frac{y^3}{3}\right]_{-1}^{1} = \frac{2}{3}(1+1)$

$=\frac{4}{3}$

Ex 2 Evaluate by Stokes' Theorem $\oint_C (e^x dx + 2y\,dy - dz)$ where C is the curve $x^2+y^2=4$, $z=2$

Solⁿ: Given we have

$$\oint_C (e^x dx + 2y\,dy - dz)$$

$$= \oint_C (e^x\hat{i} + 2y\hat{j} - \hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint_C \vec{F}\cdot d\vec{r} \quad \text{where } F = (e^x\hat{i} + 2y\hat{j} - \hat{k})$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Now

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(-1) - \frac{\partial}{\partial z}(2y)\right\} - j\left[\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial z}(e^x)\right] + k\left\{\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(e^x)\right]$$

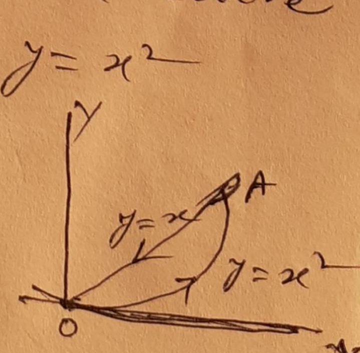
$$= 0$$

hence $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds = \iint_S 0\cdot\hat{n}\,ds = 0$

Ex ③ Verify Green's theorem in the plane for $\oint\{(xy+y^2)dx + x^2 dy\}$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$

Solⁿ: By Green's Theorem we have

$$\int_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



Here $M = xy + y^2$ & $N = x^2$

x ranges from 0 to 1

y ,, from x^2 to x

Next

Continuation of EX 3:

so $\iint_R \left\{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right\} dx\,dy$

$= \int_{x=0}^{1}\int_{y=x^2}^{x} \cancel{(2x-2y-x)}\,(2x - x - 2y)\,dx\,dy$

$= \int_{x=0}^{1}\left[\int_{y=x^2}^{x}(x-2y)\,dy\right]dx$

$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dy = \int_0^1 \left\{x^2 - x^2 - (x^3 - x^4)\right\}dx$

$= \int_0^1 (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4}$

$= \frac{-1}{20}.$

Now for the line integral along the path OA $(y = x^2)$

[St. line $y = x$ & curve $y = x^2$ cut at $(0,0)$ and $(1,1)$

x ranges from 0 to 1] Here $2x\,dx = dy$ & $y = x^2$

$\oint_C \{(xy+y^2)dx + x^2 dy\} =$

$= \int_0^1 [\{x.x^2 + (x^2)^2\}dx + x^2.2x\,dx]$

$= \int_0^1 (3x^3 + x^4)\,dx = \left[\frac{3}{4}x^4 + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5}$

$= \frac{19}{20}$

Now line integral along AO the st. line $y = x$

ranges from 1 to 0. $dy = dx$

$\oint_C \{(xy+y^2)dx + x^2 dy\}$

$= \int_1^0 [\{x.x + x^2\}dx + x^2 dx]$

$= \int_1^0 3x^2\,dx = \left[3.\frac{x^3}{3}\right]_1^0 = -1.$

required line integral is $\frac{19}{20} - 1 = -\frac{1}{20}$

So the theorem is verified.