Deptt Math. Class Notes by Amalananu Sekhar Pattanayak for GE-2 (Interdisciplinary) Department

Vector in R^n: Vectors in R^n is the collection of n-tuppl tuples of the form $(x_1, x_2 \dots x_n)$ i.e a matrix of $1 \times n$. it can also be of the form $n \times 1$: $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ ordered n tuple of Real numbers

A set $(x_1, x_2 \dots x_n)$ of real numbers arranged in the order is called n-tuples of real

Example: $(-2, 0, 4)$ is 3-tuple, $(4, 6, 8, 9)$ is 4 tuple

Note: The Collection of n vectors is called n-space and it is denoted by R^n or E^n.

Vector Space: Let F be a given field whose elements are called scalars and let V be a non-void set whose elements are vectors. The set V is called a vector space or linear space over the field F if the following axioms are satisfied.

① For any two vectors $\vec{\alpha}, \vec{\beta} \in V$, $\bar{\alpha}+\bar{\beta} \in V$ (closer prop.

② For any two vectors $\vec{\alpha}, \vec{\beta} \in V \Rightarrow \vec{\alpha}+\vec{\beta} = \vec{\beta}+\vec{\alpha}$ (Commutative

③ $\vec{\alpha}, \vec{\beta}, \vec{\gamma} \in V$, $\bar{\alpha}+(\beta+\gamma) = (\bar{\alpha}+\bar{\beta})+\gamma$ hold associative

④ for $\vec{0} \in V$. s.t $\vec{\alpha}+\bar{0} = \vec{\alpha} = \vec{0}+\vec{\alpha}$ (identity prop.

⑤ for $\alpha \in V$ there exists unique vector $-\vec{\alpha} \in V$ s.t. $\vec{\alpha}+(-\vec{\alpha}) = \vec{0}$ (additive inverse)

⑥ $a \in F$, $\vec{\alpha} \in V$, s.t. $a\vec{\alpha} \in V$ ⑦ for any a, b Scalars $\in F$, any vector $\vec{\alpha} \in E$ then $(a+b)\vec{\alpha} = a\vec{\alpha}+b\vec{\alpha}$ this is distributive of scalar ove addition

⑦ $(ab)\vec{\alpha} = a(b\vec{\alpha})$ associative prop. over multiplication

⑧ $a(\vec{\alpha}+\vec{\beta}) = a\vec{\alpha}+b\vec{\beta}$ distribution of scalar multiplication over addition.

Subspace

Defn: Let V be a vector space over a field F. Let $\vec{\alpha}_1, \vec{\alpha}_2 \cdots \alpha_r \in V$. A vector β in V is said to be linear combination of vectors $\alpha_1, \alpha_2 \cdots \alpha_r$, if β can be expressed as
$\beta = c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \cdots + c_r\vec{\alpha}_r$ where $c_1, c_2 \cdots c_r$ scalars in F.

Linear ~~defn~~ dependence and independence (L.D and L.I.)

A finite set of vectors $\{\vec{\alpha}_1, \vec{\alpha}_2 \cdots \vec{\alpha}_n\}$ of vector space V over a field F is said to be linearly dependent in V if there exists scalars $c_1, c_2 \cdots c_n$ not all zeros in F such that

$c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \cdots + c_n\vec{\alpha}_n = \vec{0}$ $\quad \cdots \cdots$ ① ①

The set is said to be linearly independent in V if equality of ① is satisfied only when
$c_1 = c_2 = \cdots = c_n = 0$. ~~attention~~

Vector Subspace

A nonempty subset W of a vector space V over a field F is called a vector ~~sp~~ subspace or linear subspace or simply subspace of V if W is a vector space over F with respect to addition and scalar multiplication.

① A nonempty subset W of V is subspace of V iff (i) $\vec{\alpha}, \vec{\beta} \in W \Rightarrow \vec{\alpha} + \vec{\beta} \in W$. (closure property addition.)
(ii) $\vec{\alpha} \in W, c \in F \Rightarrow c\vec{\alpha} \in W$

Conditions are necessary: If W is a subspace of V, all vector composition of V are satisfied by W. hence (i) & (ii) satisfied.

Condition is sufficient: F is a field therefore $-1 \in F$ if $\vec{\alpha} \in W \Rightarrow -1\vec{\alpha} \in W \Rightarrow -\vec{\alpha} \in W$.

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By (i) $\vec{\alpha}+(-\vec{\alpha}) \in W \Rightarrow \vec{\phi} \in W$, $\vec{\phi}$ = null vector

All the elements of W ~~are~~ satisfy vector space condition and so W is a subspace of V.

NB (i) If W is a subspace of a vector space V over the field F. Then for all $a_1, a_2 \ldots a_n \in F$ and for all $\vec{\alpha}_1, \vec{\alpha}_2 \ldots \vec{\alpha}_n \in W$, then

$$a_1\vec{\alpha}_1 + a_2\vec{\alpha}_2 + \cdots + a_n\vec{\alpha}_n \in W.$$ for each +ve integer n.

Ex. If a_1, a_2, a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of F s.t. $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a linear vector subspace of $V_3(F)$

Solⁿ: Let $\vec{\alpha} = (x_1, x_2, x_3)$, $\vec{\beta} = (y_1, y_2, y_3) \in W$ then

$$a_1x_1 + a_2x_2 + a_3x_3 = 0, \quad a_1y_1 + a_2y_2 + a_3y_3 = 0$$

Now $a\vec{\alpha} + b\vec{\beta} = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$ for $a, b \in F$.

and
$$a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$$
$$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3) = 0$$

$\therefore a\vec{\alpha} + b\vec{\beta} \in W$

Hence W is a linear vector subspace of $V_3(F)$

NB. The intersection of any two subspace of a vector space V(F) is also a subspace.

Proof: Let W_1, W_2 any two subspace of V(F). Since $\vec{\phi} \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is not empty.

Case-I Let $W_1 \cap W_2 = \{\vec{\phi}\}$ Then $W_1 \cap W_2$ is the trivial subspace of V(F).

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Case II. Let $W_1 \cap W_2 \neq \{\vec{\phi}\}$ and $\vec{\alpha}, \vec{\beta} \in W_1 \cap W_2$ and $a, b \in F$. Then:

$\vec{\alpha} \in W_1$ and $\vec{\alpha} \in W_2$ as $\vec{\alpha} \in W_1 \cap W_2$

$\vec{\beta} \in W_1$ and $\vec{\beta} \in W_2$ as $\vec{\beta} \in W_1 \cap W_2$.

$\vec{\alpha}, \vec{\beta} \in W_1 \Rightarrow a\vec{\alpha} + b\vec{\beta} \in W_1$ as W_1 is a Subspace

$\vec{\alpha}, \vec{\beta} \in W_2 \Rightarrow a\vec{\alpha} + b\vec{\beta} \in W_2$ as W_2 " "

$\therefore$ $a\vec{\alpha} + b\vec{\beta} \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of $V(F)$.

N.B (I) A set of vectors which contains a null vector is L.D. (linear dependence).

(ii) The singleton $\{\vec{\alpha}\}$ is L.I. if $\vec{\alpha} \neq \vec{0}$.

(iii) A Subset of L.D. set of vectors is L.D.

For (I) Let the set be $\{\vec{\alpha_1}, \vec{\alpha_2}, \cdots \vec{\alpha_k}, \vec{\phi}\}$ in $V(F)$ and $c_1, c_2 \cdots c_{k+1} \in F$.

The relation $c_1\vec{\alpha_1} + c_2\vec{\alpha_2} + \cdots c_k\vec{\alpha_k} + c_{k+1}\vec{\phi} = 0$ holds if $c_1 = c_2 = \cdots = c_k = 0$ but $c_{k+1} \neq 0$. Thus set is L.D.

(ii) Here $c\vec{\alpha} = \vec{\phi} \Rightarrow c = 0$, Hence the result:

(iii) Let $\alpha_i \in V_n(F)$, $c_i \in F$ and $(\vec{\alpha_1}, \vec{\alpha_2} \cdots \vec{\alpha_p})$ be L.I. Then $\sum_{i=1}^{p} c_i \alpha_i = \vec{\phi}$ holds only all $c_i = 0$.

Let $\{\vec{\alpha_1}, \vec{\alpha_2} \cdots \vec{\alpha_k}\}$ be a subset $1 \leq k \leq p$.

~~At~~ $\lambda_1\vec{\alpha_1} + \lambda_2\vec{\alpha_2} + \cdots + \lambda_k\vec{\alpha_k} = \vec{\phi}$ for $\lambda_1, \lambda_2 \cdots \lambda_k \in F$.

Can be written as $\lambda_1\vec{\alpha_1} + \lambda_2\vec{\alpha_2} + \cdots + \lambda_k\vec{\alpha_k} + 0\,\vec{\alpha_{k+1}} + \cdots + 0\alpha_p = \vec{\phi}$

Since $\{\vec{\alpha_1}, \vec{\alpha_2} \cdots \vec{\alpha_p}\}$ is L.I.

$\therefore$ $\lambda_1 = \lambda_2 = \cdots \lambda_k = 0$.

Hence the result follows.

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GE-2 Dimention

Basis:- A subset S of vector space V(F) is said to the basis of V(F) if ① S is linearly independe and ii) any vector in V other than that of S is in a linear combination of the vectors of S [i.e. $L(S) = V$]

Dimension: The number of elements in a basis of the vector space V(F) is called the dimension (or rank) of V(F) and is denoted by dim V.

Note I. In a basis set no element is null vector

② If the number of vectors in a basis of V(F) is finite, then V is said to be finite dimensional or finitely generated.

③ Null vector has no basis and its dimension is zero.

Dimension of a sub space

Th. Every subspace W of a finite dimensional vector space V(F) of dimension n is also a finite dimensional vector space of dimension m s.t. ~~W~~ $m \le n$ i.e $\dim W \le \dim V$.

Also $V = W$ iff $m = n$.

Proof:- Let $B = \{\vec{\alpha}_1, \vec{\alpha}_2 \cdots \vec{\alpha}_n\}$, $B_1 = \{\vec{\beta}_1, \vec{\beta}_2 \cdots \beta_m\}$ be the bases of V and W respectively.

① Let $m > n$. Each vector of B_1 is also a vector of V. Then $\vec{\alpha}_1, \vec{\alpha}_2 \cdots \vec{\alpha}_n$ can be replaced by n vectors of B_1 (as $m > n$), If B' is the new set after replacement then $B' = \{\vec{\beta}_1, \vec{\beta}_2 \cdots \beta_n\}$ and $B_1 = \{\vec{\beta}_1, \vec{\beta}_2 \cdots \vec{\beta}_n, \vec{\beta}_{n+1}, \cdots \beta_m\}$. B' is a basis of V. Therefore each of $\vec{\beta}_{n+1}, \vec{\beta}_{n+2} \cdots \beta_m$ is dependent upon B'.

It contradicts the fact that B_1 is linearly independent. Hence m cannot be greater than n. Thus $m \le n$. Hence the result.

Th. ② If W_1 and W_2 are two subspaces of a finite dimensional vector space $V(F)$ then

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Sum If $W = \{(x,y,z) \in R^3 : x - 4y + 3z = 0\}$ [Show that it is a sol^n of R^3] Find dimension of W.

Sol^n: Let $\vec{u} = (a,b,c) \in W$ then $a - 4b + 3c = 0$

$\therefore \vec{u} = (4b-3c, b, c) = b(4,1,0) + c(-3,0,1)$

Let $\vec{\alpha} = (4,1,0)$, $\beta = (-3,0,1)$, $\alpha, \beta \in W$

Again $\vec{u} = b\vec{\alpha} + c\vec{\beta}$, $b, c \in R$.

Therefore W is a subspace of R^3.

$$W = L(\{\vec{\alpha}, \vec{\beta}\})$$

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Let $c_1\vec{\alpha} + c_2\vec{\beta} = \vec{\theta}$, $c_1, c_2 \in R$

it implies that $4c_1 - 3c_2 = 0$

i.e. $c_1(4,1,0) + c_2(-3,0,1) = \vec{0}$

$\therefore 4c_1 - 3c_2 = 0$, $c_1 = 0$ and $c_2 = 0$.

$\Rightarrow \vec{\alpha}, \vec{\beta}$ linearly independent

so basis of W, for ~~two~~ numbers of vectors is 2 so $\dim(W) = 2$