CHAPTER 19

Laplace Transformation and its Application

19.1 Introduction

The operation of differentiating functions is a transformation from functions $F(t)$ to function $F'(t)$. If the operation is represented by the symbol D, the transformation can be written as

$$D\{F(t)\} = F'(t)$$

The function $F'(t)$ is the image of $F(t)$ under this transformation; the function $2t$, for example, is the image of the function t^2.

Another transformation of functions is that of integration.

$$I\{F(t)\} = \int_0^x F(t)\, dt$$

The result of this operation is a function $f(x)$, the image of $F(t)$ under this transformation. In each of the above examples an inverse transformation exists, provided some sufficient conditions are being satisfied by $F(t)$.

A transformation $T\{F(t)\}$ is said to be linear if for every pair of function $F_1(t)$ and $F_2(t)$ and for each pair of constant C_1 and C_2, it satisfies the relation

$$T\{C_1F_1(t) + C_2F_2(t)\} = C_1T\{F_1(t)\} + C_2T\{F_2(t)\}$$

Thus in a linear transformation, the transform of a linear combination of two functions is the same linear combination of the transforms of those functions.

Linear integral transformations of the functions of t defined on a finite or infinite interval $a < t < b$ are sometimes very useful in solving problems in differential equations.

Let $K(t, s)$ denotes some prescribed function of a variable t and a parameter s. A general linear integral transformation of a function $F(t)$ with respect to $K(t, s)$, called kernel, is represented by the equation

$$T\{F(t)\} = \int_a^b K(t, s)\, F(t)\, dt$$

It represent a function $f(s)$ known as the image or transform of the function $F(t)$. It is seen that with certain kernel $K(t, s)$ of the transformation when applied to a prescribed linear differential equation in $F(t)$, it changes its form into an algebraic expression $f(s)$ that involves certain boundary values of the function $F(t)$. Consequently, many problems in ordinary differential equations transform into algebraic

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problems taking the image of the unknown functions. If an inverse transformation is possible, the solution of the original problem can be determined. Boundary value problems in partial differential equation can be simplified in a similar way.

The Laplace transform is an integral transform. It is effective for the solution of many problems arising in science and engineering. Laplace transformation converts an ordinary differential equation with some given initial conditions into an algebraic equation. Finally using inverse Laplace transformation, we recover the original function. Considering its usefulness we shall first consider Laplace transform with its properties and then discuss the method of solution of differential equation by the application of this transform.

19.2 Laplace Transform

■ **Definition :** If a function $F(t)$, defined for all positive values of the variable t, is multiplied by e^{-st} and integrated with respect to t from zero to infinity, a new function $f(s)$ of the parameter s (> 0) is obtained i.e.,

$$f(s) = \int_0^\infty e^{-st} F(t)dt$$

This operation on a function $F(t)$ is called the Laplace transformation. Using symbol, we may write

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)dt.$$

From elementary property of integral, it is easy to see that Laplace transformation is a linear operation. Also it is obvious that the Laplace transformation may not be convergent for all values of the parameter s.

The inverse Laplace transform, denoted by L^{-1}, will be defined by

$$L^{-1}\{f(s)\} = F(t)$$

if $f(s)$ be the Laplace transform of $F(t)$.

19.3 Laplace Transform of some Elementary Functions

1. If $F(t) = A$, a constant, then

$$L\{A\} = \int_0^\infty Ae^{-st}dt = \left[-\frac{Ae^{-st}}{s}\right]_0^\infty = \frac{A}{s},\ s > 0$$

2. $F(t) = t$, then

$$L\{t\} = \int_0^\infty te^{-st}dt = \frac{1}{s^2}$$

3. $F(t) = t^n$, then

$$L\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}},$$ when n is a positive integer.

$$= \frac{\Gamma(n+1)}{s^{n+1}},\ s > 0$$ when n is positive but not necessarily an integer.

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4. $F(t) = e^{kt}$, then

$$L\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt}\,dt = \left[\frac{-e^{-(s-k)t}}{s-k}\right]_0^\infty = \frac{1}{s-k},\ s > k$$

5. $F(t) = \cos kt$, then

$$L\{\cos kt\} = \int_0^\infty e^{-st} \cos kt\,dt = \left[\frac{e^{-st}}{s^2+k^2}(-s\cos kt + k\sin kt)\right]_0^\infty = \frac{s}{s^2+k^2},\ s > 0$$

Similar, $L\{\sin kt\} = \dfrac{k}{s^2+k^2}$

6. $L\{\cosh kt\} = L\left\{\frac{1}{2}(e^{kt}+e^{-kt})\right\} = \frac{1}{2}\left(\frac{1}{s-k}+\frac{1}{s+k}\right) = \frac{s}{s^2-k^2},\ s > |k|$

Similarly, $L\{\sinh kt\} = \dfrac{k}{s^2-k^2},\ s > |k|$.

19.4 The Inverse Laplace Transform of some Simple Functions

1. $L^{-1}\left(\frac{1}{s}\right) = 1$, since $L(1) = \frac{1}{s}$

2. $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$, since $L\{t^n\} = \frac{n!}{s^{n+1}}$

3. $L^{-1}\left(\frac{1}{s-k}\right) = e^{kt}$, since $L\{e^{kt}\} = \frac{1}{s-k}$

4. $L^{-1}\left(\frac{1}{s^2-k^2}\right) = \frac{1}{k}\sin kt$, since $L\{\sin kt\} = \frac{k}{s^2+k^2}$

5. $L^{-1}\left(\frac{s}{s^2+k^2}\right) = \cos kt$, since $L\{\cos kt\} = \frac{s}{s^2+k^2}$

6. $L^{-1}\left(\frac{1}{s^2-k^2}\right) = \frac{1}{a}\sinh kt$, since $L\{\sinh kt\} = \frac{k}{s^2-k^2}$

7. $L^{-1}\left(\frac{s}{s^2-k^2}\right) = \cosh kt$, since $L\{\cosh kt\} = \frac{s}{s^2-k^2}$

8. $L^{-1}\left\{\frac{2ks}{(s^2+k^2)^2}\right\} = t\sin kt$

9. $L^{-1}\left\{\frac{s^2-k^2}{(s^2+k^2)^2}\right\} = t\cos kt$

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10. $L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^3}$ (sin kt – kt cos kt)

11. $L^{-1}\left\{\frac{1}{(s+k)^2}\right\} = te^{-kt}$

12. $L^{-1}\left\{\frac{k}{(s+b)^2+k^2}\right\} = e^{-bt}\sin kt$

13. $L^{-1}\left\{\frac{s+b}{(s+b)^2+k^2}\right\} = e^{-bt}\cos kt$

14. $L^{-1}\left\{\frac{s+b}{(s+b)^2-k^2}\right\} = e^{-bt}\cosh kt$

15. $L^{-1}\left\{\frac{k}{(s+b)^2-a^2}\right\} = e^{-bt}\sinh kt$

16. $L^{-1}\left\{\frac{1}{(s+b)^n}\right\} = e^{-bt}\frac{t^{n-1}}{(n-1)!}$

17. $L^{-1}\left(\frac{1}{s}\right) = H(t)$

18. $L^{-1}\left(\frac{e^{-ks}}{s}\right) = H(t-k)$

19. $L^{-1}(1) = \delta(t)$

20. $L^{-1}(e^{-ks}) = \delta(t-k)$

21. $L^{-1}\left(\frac{1}{s^2+4}\right) = \sin t\cos t$ for $s > 0$

22. $L^{-1}\left\{\frac{s^2+2}{4(s^2+4)}\right\} = \cos^2 t$ for $s > 0$

19.5 Piece-wise (or Sectionally Continuous) Functions and Functions of Exponential Order

A function $F(t)$ is *piece-wise* (or *sectionally*) *continuous* on a finite interval $a \le t \le b$, if it is such that the interval can be subdivided into a finite number of subintervals like $c \le t \le d$, such that in each such subinterval

1) $F(t)$ is continuous in the open interval $c < t < d$

2) $F(t)$ approaches a finite limit as t approaches each end point from within the interval

i.e., $\underset{t \to c+}{Lt} F(t)$ and $\underset{t \to d-}{Lt} F(t)$ exist.

A function $F(t)$ is of *exponential order* as t tends to infinity, provided some constant α exists such that the product $e^{-\alpha t}|F(t)|$ is bounded for all t grater than some finite number T. Thus $|F(t)|$ does not grow more rapidly that $Me^{\alpha t}$ as $t \to \infty$ where M is some positive constant. This is expressed by saying that $F(t)$ is of the order of $e^{\alpha t}$.

Functions having the above two properties are called *function of class A*.

19.6 Sufficient Conditions for the Existence of Laplace Transform

Theorem : If $F(t)$ is a function of class A, $L\{F(t)\}$ exists i.e., If $F(t)$ be a piece-wise continuous function in the interval $[0, T]$ for every positive T and let $F(t)$ be of exponential order as $t \to \infty$ for some $a > 0$, then the Laplace transform of F exists for $p > q$.

Proof : Since $F(t)$ is a piece-wise continuous and of exponential order, we have

$$|L\{F(t)\}| = \left|\int_0^\infty e^{-st} F(t) dt\right| \le \int_0^\infty e^{-st} |F(t)|\, dt \le \int_0^\infty e^{-st} \cdot Me^{at} dt,$$ M being some constants.

The right hand side is equal to

$$M \int_0^\infty e^{-(s-a)t} dt = \frac{M}{s-a},\ s > a$$

Thus $\int_0^\infty e^{-st} F(t)\, dt$ exists for $s > a$.

19.7 Properties of Laplace Transform and its Inverse

(i) Linearity property of Laplace transforms :

If c_1 and c_2 be constants, then

$$L\{c_1F_1(t) + c_2F_2(t)\} = c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$$

Proof : We know, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

So,
$$L\{c_1F_1(t) + c_2F_2(t)\} = \int_0^\infty e^{-st} \{c_1F_1(t) + c_2F_2(t)\}\, dt$$
$$= c_1 \int_0^\infty e^{-st} F_1(t)\, dt + c_2 \int_0^\infty e^{-st} F_2(t)\, dt$$
$$= c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$$

Similarly if $f_1(s)$ and $f_2(s)$ be the Laplace transforms of the function $F_1(t)$ and $F_2(t)$ respectively and q_1, q_2 be constants, then it can be shown that

$$L^{-1}\{q_1 f_1(s) + q_2 f_2(s)\} = q_1L^{-1}\{f_1(s)\} + q_2L^{-1}\{f_2(s)\}$$

Thus the Laplace transform and its inverse are linear.

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■ **(ii) First shifting (or first translation) theorem :**

If $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} = f(s-a)$

where a is any real or complex constant.

Proof : From definition we have,

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Replacing s by $(s-a)$ on both sides we get

$$f(s-a) = \int_0^\infty e^{-(s-a)t} F(t) dt$$

$$= \int_0^\infty e^{-st} \{e^{at} F(t)\}\, dt$$

$$= L\{e^{at} F(t)\}$$

Evedently, $L^{-1}\{f(s-a)\} = e^{at} F(t)$.

■ **(iii) Change of scale property**

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof : By definition, we have

$$L\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}y} F(y)\, dy, \text{ by putting } at = y$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t} F(t)\, dt$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right).$$

19.8 Laplace Transform of the Integrals

If $f(s)$ be the Laplace transform of $F(t)$, then

$$L\left\{\int_0^t F(T)\, dT\right\} = \frac{f(s)}{s} = \frac{1}{s}L\{F(t)\}$$

Proof : We have

$$L\left\{\int_0^t F(T)\, dT\right\} = \int_0^\infty \left\{\int_0^t F(T)\, dT\right\} e^{-st}\, dt$$

$$= \left[-\frac{e^{-st}}{s}\int_0^t F(T)\, dT\right]_0^\infty + \frac{1}{s}\int_0^\infty F(t)\, e^{-st} dt$$

$$= \frac{f(s)}{s}, \text{ since } \int_0^t F(T)\, dT \text{ is an exponential order}$$

$$= \frac{1}{s} L\{F(t)\}$$

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19.9 Convolution Theorem

The convolution $F * G$ of the functions $F(t)$ and $G(t)$ is defined as

$$F(t) * G(t) = \int_0^t F(\tau)\, G(t-\tau)\, d\tau$$

The convolution satisfies the following properties :

(a) **Commutative property** i.e., $F(t) * G(t) = G(t) * F(t)$

(b) **Associative property** i.e., $F(t) * [G(t) * H(t)] = [F(t) * G(t)] * H(t)$

(c) **Distributive property** i.e., $F(t) * [G(t) + H(t)] = [F(t) * G(t)] + [F(t) * H(t)]$

■ **Theorem** : If $f(s)$ and $g(s)$ are the transforms of two functions $F(t)$ and $G(t)$, that are sectionally (piece-wise) continuous on the interval $0 \le t \le T$ for any finite $T \ge 0$ and are of the order of $e^{\alpha t}$ as $t \to \infty$, then the transform of the convolution $F(t) * G(t)$ exists when $s > \alpha$ and it is equal to $f(s)\, g(s)$. Thus the inverse transform of the product $f(s)\, g(s)$ is given by the formula

$$L^{-1}\{ f(s)\, g(s)\} = F(t) * G(t).$$

19.10 Laplace Transform of a Function Multiplied by the Integral Power of t

■ **Theorem** : If $F(t)$ be a function of class A and if $L\{F(t)\} = f(s)$, then

$$L\{tF(t)\} = -f'(s)$$

Proof : From definition we have $f(s) = \int_0^\infty e^{-st} F(t)\, dt$

Since $F(t)$ belongs to class A, Differentiating under the sign of integral by Leibnitz rule, we get

$$\frac{d}{ds} f(s) = \int_0^\infty (-t) e^{-st} F(t) dt = -\int_0^\infty e^{-st} \{tF(t)\}\, dt$$

or, $f'(s) = -L\{tF(t)\}$

or, $L\{tF(t)\} = -f'(s)$

Note : Leibnitz's rule for differentiation under the integral sign.

Let a and b be constants. Then

$$f(s) = \int_a^b F(s,t)\}\, dt$$

$$\text{or,} \quad \frac{d}{ds} f(s) = \int_a^b \frac{\partial}{\partial s} F(s,t)\}\, dt$$

In general, If $F(t)$ is a function of class A and if $L\{F(t)\} = f(s)$ then

$$L\{t^n F(t) = (-1)^n \frac{d^n f(s)}{ds^n},\; n = 1, 2, 3, \ldots\ldots$$

Now the theory can be proved by the method of induction.

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19.11 Laplace Transform of a Function Divided by t

■ **Theorem** : If $L\{F(t)\} = f(s)$, then

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds,$$ provided the integral exists.

Proof : From definition, $f(s) = \int_0^\infty e^{-st} F(t)dt.$

Integrating with respect to s from s = s to s = ∞, we get

$$\int_s^\infty f(s)ds = \int_s^\infty \left\{\int_0^\infty e^{-st} F(t)dt\right\} ds$$

Since s and t are independent variables, the order of integration in the double integral of right hand side can be interchanged.

Thus, $$\int_s^\infty f(s)ds = \int_0^\infty \left\{\int_s^\infty e^{-st} ds\right\} F(t)dt = \int_0^\infty \left[-\frac{e^{-st}}{t}\right]_s^\infty F(t)dt$$

$$= \int_0^\infty e^{-st} \frac{F(t)}{t} dt$$

Hence, $$\int_s^\infty f(s)\,ds = L\left\{\frac{F(t)}{t}\right\}$$

■ **Corollary** : If $L\{F(t)\} = f(s)$, then

$$\int_0^\infty \frac{F(t)}{t} dt = \int_0^\infty f(s)\,ds,$$ provided the integral converges.

Proof : From defination

$$L\left\{\frac{F(t)}{t}\right\} = \int_0^\infty e^{-st} \frac{F(t)}{t} dt = \int_s^\infty f(s)ds = \int_s^\infty f(x)dx$$

i.e., $$\int_0^\infty e^{-st} \frac{F(t)}{t} dt = \int_s^0 f(x)dx + \int_0^\infty f(x)dx = -\int_0^s f(x)dx + \int_0^\infty f(x)dx$$

Taking limits of both sides as $s \to 0+$ and assining that the integral converges, we get

$$\int_0^\infty \frac{F(t)}{t} dt = \int_0^\infty f(x)dx = \int_0^\infty f(s)\,ds$$

19.12 Laplace Transform of Two Special Functions

1. The Error Function : The error function is denoted as *'erf'* or E and defined as

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad \text{... (1)}$$

The complement of error function *erf*(x) is denoted as *'erfc'* and defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - erf(x) \qquad \text{... (2)}$$

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