Semester - VI

Course Type - DSE-A

Course Title - DSE-A Mathematics Modeling

Topic - Laplace Transformation

References - Dr. Arup Mukherjee & Dr. Nala Kumar Bej books.

Date - 29.03.2020

— Shambhu Nath Acharya

▫ Now we will prove that

$$L\{erf\sqrt{t}\} = \frac{1}{s\sqrt{s+1}} \quad \text{... (3)}$$

We know that, $L\left(t^{-\frac{1}{2}}\right) = \int_0^\infty e^{-st} t^{-\frac{1}{2}}\, dt$

Putting $\sqrt{t} = \frac{1}{\sqrt{s}} y$, $\frac{1}{2\sqrt{t}} dt = \frac{1}{\sqrt{s}} dy$,

$$\int_0^\infty e^{-st} t^{-\frac{1}{2}}\, dt = \frac{2}{\sqrt{s}} \int_0^\infty e^{-y^2} dy = \frac{2}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}}$$

Therefore, $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

or, $L\left\{\frac{e^{-t}}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s+1}}$ [by shifting theorem]

Hence, $\frac{1}{s\sqrt{s+1}} = L\{t^0\} \cdot L\left\{\frac{e^{-t}}{\sqrt{\pi t}}\right\}$

$$= L\left\{\int_0^t (t-\tau)^0 \frac{e^{-\tau}}{\sqrt{\pi\tau}} d\tau\right\} \text{ [using convolution theorem]}$$

$$= L\left\{\int_0^t 1 \cdot \frac{e^{-\tau}}{\sqrt{\pi\tau}} d\tau\right\}$$

Let $\tau = u^2$, then the integral on right hand side

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du = erf\sqrt{t}$$

Thus we find that

$$L\{erf\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$$

Similarly, it can be proved that for complementary error function *erfcx*,

$$L\left\{erfc\left(\frac{k}{2\sqrt{t}}\right)\right\} = \frac{1}{s} e^{-k\sqrt{s}} \quad \text{... (4)}$$

2. The Bessel function : The Bessel function of order n is denoted by $J_n(x)$ and is defined by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

❑ **Now we will prove that**

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$

From definition, we have

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+1)}\left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r}}{2^{2r}(r!)^2}$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \ldots$$

Therefore, $L\{J_0(t)\} = L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2.4^2}L\{t^4\} - \frac{1}{2^2.4^2.6^2}L\{t^6\} + \ldots$

$$= \frac{1}{s} - \frac{1}{2^2}.\frac{2!}{s^3} + \frac{1}{2^2.4^2}.\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}.\frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}.\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3 + \ldots\right\}$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+s^2}}$$

So, $L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$

19.13 Worked Out Examples

Example 1. Find $L\{t^2 + 1)^2\}$.

Solution :

We have

$$L\{t^2 + 1)^2\} = L\{t^4 + 2t^2 + 1\} = L\{t^4\} + 2L\{t^2\} + L\{1\}$$

$$= \frac{4!}{s^5} + 2.\frac{2}{s^3} + \frac{1}{s},\ s > 0 \quad \left[\text{since } L\{t^n\} = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{24 + 2s^2 + s^4}{s^5},\ s > 0$$

Example 2. Find $L\left\{\frac{e^{at}-1}{a}\right\}$.

Solution :

We have

$$L\left\{\frac{e^{at}-1}{a}\right\} = \frac{1}{a}L\{e^{at} - 1\} = \frac{1}{a}[L\{e^{at}\} - L\{1\}]$$

$$= \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{s(s-a)},\ \text{if } s > a \text{ and } s > 0$$

Example 3. Find $L\{\sin^3 2t\}$.

Solution :

We have

$$L\{\sin^3 2t\} = L\left\{\frac{1}{4}(3\sin 2t - \sin 6t)\right\} \quad [\text{since, } \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$$

$$= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4}\cdot\frac{2}{s^2+2^2} - \frac{1}{4}\cdot\frac{6}{s^2+6^2},\ s > 0$$

$$= \frac{48}{(s^2+4)(s^2+36)},\ s > 0$$

Example 4. Find $L\{F(t)\}$, where

$$F(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2} \\ t-1, & \frac{1}{2} < t \le 1 \\ 0, & t \ge 1 \end{cases}$$

Solution :

From definition

$$\int_0^\infty e^{-st}F(t)\,dt = \int_0^{\frac{1}{2}} e^{-st}F(t)\,dt + \int_{\frac{1}{2}}^{1} e^{-st}F(t)\,dt + \int_1^\infty e^{-st}F(t)\,dt$$

$$= \int_0^{\frac{1}{2}} e^{-st}t\,dt + \int_{\frac{1}{2}}^{1} e^{-st}(t-1)\,dt + \int_1^\infty e^{-st}\,0\,dt$$

$$= \left[\frac{te^{-st}}{-s}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\left[-\frac{e^{-st}}{s}\right]dt + \left[\frac{(t-1)e^{-st}}{-s}\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1}\left[-\frac{e^{-st}}{s}\right]dt$$

$$= -\frac{1}{2s}e^{-\frac{s}{2}} + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_0^{\frac{1}{2}} - \frac{1}{2s}e^{-\frac{s}{2}} + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_{\frac{1}{2}}^{1}$$

$$= -\frac{1}{s}e^{-\frac{s}{2}} + \frac{1}{s^2}e^{-\frac{s}{2}} + \frac{1}{s^2} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}e^{-\frac{s}{2}}$$

$$= -\frac{1}{s}e^{-\frac{s}{2}} + \frac{1}{s^2}(1-e^{-s})$$

Example 5. Show that $L\{(1+te^{-t})^3\} = \dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$.

Solution :

We have

$$L\{(1+te^{-t})^3\} = L\{1 + 3te^{-t} + 3t^2e^{-2t} + t^3e^{-3t}\} \quad \text{... (1)}$$

$$= L\{1\} + 3L\{te^{-t}\} + 3L\{t^2e^{-2t}\} + L\{t^3e^{-3t}\}$$

Now $L\{t\} = \frac{1!}{s^2}, \; L\{t^2\} = \frac{2!}{s^3}, \; L\{t^3\} = \frac{3!}{s^4}$

Therefore, by first shifting theorem, we have

$$L\{e^{-t}t\} = \frac{1!}{(s+1)^2}, \; L\{e^{-2t}t^2\} = \frac{2!}{(s+2)^3} \text{ and } L\{e^{-3t}t^3\} = \frac{3!}{(s+3)^4}$$

putting the value in (1), we have

$$L\{(1 + te^{-t})^3\} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Example 6. If $L\{F(t)\} = \frac{(s^2 - s + 1)}{(2s+1)^2(s-1)}$, prove that $L\{F(2t)\} = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$.

Solution :

We have $L\{F(t)\} = f(s) = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$

Now, by the change of scale property we have

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{s}{2}\right) = \frac{1}{2} \cdot \frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(2 \cdot \frac{s}{2} + 1\right)^2 \left(\frac{s}{2} - 1\right)} = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$$

Example 7. Find $L\{t \sin at\}$.

Solution :

We have $L\{\sin at\} = \frac{a}{s^2 + a^2}, \; s > 0$

Hence
$$L\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right)$$
$$= -a \frac{d}{ds}(s^2 + a^2)^{-1} = \frac{2as}{(s^2 + a^2)^2}$$

Example 8. Find $L\{t^3 \cos t\}$.

Solution :

We have $L\{\cos t\} = \frac{s}{s^2 + 1}, \; s > 0$

Therefore,
$$L\{t^3 \cos t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{s}{s^2 + 1}\right)$$
$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{s}{s^2 + 1}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{(s^2+1)-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{s^2-1}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)^2 \,.\, 2s-(s^2-1)\,.\,2(s^2+1)\,.\,2s}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\cdot\frac{6s-2s^3}{(s^2+1)^3} = \frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

Example 9. Find the Laplace transform of $\frac{\sin at}{t}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Solution :

We know,

$$L\{\sin at\} = \frac{9}{s^2+9^2} = f(s), \text{ (say)} \qquad \dots (1)$$

Therefore,
$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} f(s)\,ds = \int_s^{\infty}\frac{a}{s^2+q^2}\,ds, \text{ [using (1)]}$$

$$= a\cdot\frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^{\infty} = \tan^{-1}\infty - \tan^{-1}\frac{s}{a}$$

$$= \frac{\pi}{2}-\tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

Now,
$$L\{\cos at\} = \frac{s}{s^2+a^2} = g(s), \text{ (say)} \qquad \dots (2)$$

So,
$$L\left\{\frac{\cos at}{t}\right\} = \int_s^{\infty} g(s)\,ds = \int_s^{\infty}\frac{s}{s^2+a^2}\,ds \text{ [using (2)]}$$

$$= \left[\frac{1}{2}\log(s^2+a^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\lim_{s\to\infty}\log(s^2+a^2)-\frac{1}{2}\log(s^2+a^2)$$

which does not exists since

$$\lim_{s\to\infty}\log(s^2+a^2) \text{ does not exits.}$$

Hence $L\left\{\frac{\cos at}{t}\right\}$ does not exist.

Example 10. Find $L^{-1}\left\{\frac{1}{s(s+5)}\right\}$.

Solution :

We have

$$L^{-1}\left\{\frac{1}{s(s+5)}\right\} = \frac{1}{5}L^{-1}\left[\frac{1}{s}-\frac{1}{s+5}\right]$$

$$= \frac{1}{5}\left[1-e^{-5t}\right] \quad \left[\text{since } L(1)=\frac{1}{s} \text{ and } L(e^{-5t})=\frac{1}{s+5}\right]$$

Example 11. Find $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$.

Solution :

We have

$$L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\} = L^{-1}\left\{\frac{3}{4}\cdot\frac{s}{s^2+\left(\frac{5}{2}\right)^2}-2\cdot\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right\}$$

$$= \frac{3}{4}L^{-1}\left\{\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right\}-2L^{-1}\left\{\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right\}$$

$$= \frac{3}{4}\cos\frac{5t}{2}-2\cdot\frac{1}{\frac{5}{2}}\cdot\sin\frac{5t}{2}$$

$$= \frac{3}{4}\cos\frac{5t}{2}-\frac{4}{5}\sin\frac{5t}{2}$$

Example 12. Evaluate $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$.

Solution :

We have,

$$\frac{1}{(s^2+4)(s+1)^2} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4}$$

or, $1 = A(s+1)(s^2+4) + B(s^2+4) + (Cs+D)(s+1)^2$ ... (1)

Putting $s = -1$ in (1) we get, $B = \frac{1}{5}$.

Again, equating the coefficients of s^3, s^2 and constants from both side of (1) and simplifying, we get, $A = \frac{2}{25}$, $C = -\frac{2}{25}$ and $D = -\frac{3}{25}$.

So, $$\frac{1}{(s^2+4)(s+1)^2} = \frac{2}{25(s+1)} + \frac{1}{5(s+1)^2} - \frac{2s+3}{25(s^2+4)}$$

Therefore, $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$

$$= \frac{2}{25}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \frac{2}{25}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{3}{25}L^{-1}\left\{\frac{1}{s^2+2^2}\right\}.$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{2}{25}\cos 2t - \frac{3}{25}\cdot\frac{\sin 2t}{2}$$

$$= \frac{2}{25}e^{-t} + \frac{1}{5}te^{-t} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t$$

Example 13. Use the convolution theorem to evaluate

(i) $L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\}$ **(ii)** $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$.

Solution :

(i) Let $f(s) = \frac{1}{s+1}$ and $g(s) = \frac{1}{s-1}$

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$ ... (1)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\left(\frac{1}{s-1}\right) = e^{t}$ ... (2)

Now using the convolution theorem, we have

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u)\,G(t-u)du$$

or, $L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} = \int_0^t e^{-u}e^{t-u}\,du$, [using (1) and (2)]

$$= e^t\int_0^t e^{-2u}du = e^t\left[-\frac{1}{2}e^{-2u}\right]_0^t$$

$$= -\frac{1}{2}e^t(e^{-2t}-1) = \frac{1}{2}(e^t - e^{-t})$$

(ii) $f(s) = \frac{1}{s+1}$ and $g(s) = \frac{1}{s^2+1}$ ... (1)

Therefore, $F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$