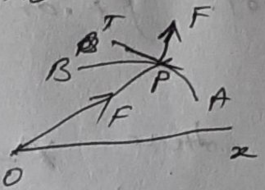
Notes on Math Hons Sem -4 Paper C-9T
Unit-3. (Line integral,

Amalananda Sekhar Pattanayak.

Let $\vec{F}(x,y,z)$ be a vector function and a curve AB.
Line integral of vector function $\vec{F}$ along the curve AB is defined as Line integral of the component of $\vec{F}$ along the tangent to the curve AB.



Components of $\vec{F}$ along tangent is $\vec{PT}$ at P.
$=$ Dot Product of $\vec{F}$ and unit vector along PT
$= \vec{F}\cdot\frac{d\vec{r}}{ds}$ ($\frac{dr}{ds}$ is unit vector along PT.)

∴ Line integral $= \sum \vec{F}\cdot\frac{d\vec{r}}{ds}$ from A to B.
$= \int\left(\vec{F}\cdot\frac{dr}{ds}\right)ds = \int_C \vec{F}\cdot dr$.

N.B. When the path of integration is closed curve then notion of integration is $\oint$ in place of $\int$.

N.B. work done $= \int_A^B \vec{F}\cdot d\vec{r}$, $\vec{F}$ is variable force along AB.

Ex① If a force $\vec{F} = 2x^2y\,i + 3xy\,j$ displaces a particle in xy plane from (0,0) to (1,4) along curve $y = 4x^2$. Find the work done (Find the line integral along the path or Evaluate it

Sol^n: $\int F\cdot dr$, $\vec{r} = xi + yj + zk \Rightarrow d\vec{r} = dx\,i + dy\,j + dz\,k$ [for xy, z plane].

$= \int_C (2x^2y\,i + 3xy\,j)\cdot(i\,dx + j\,dy) = \int (2x^2y\,i + 3xy\,j)\cdot(i\,dx + j\,dy + k\,dz)$

$= \int (2x^2y\,dx + 3xy)\,dy$ Putting value of y and dy [$y = 4x^2$, $dy = 8x\,dx$ and $x \rightarrow 0$ to 1],

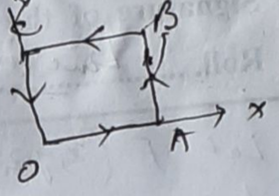
$= \int_0^1 [(2x^2)\cdot 4x^2dx + 3x\cdot 4x^2dx] = \int_0^1 (8x^2 + 96x^2)\,dx = 104\left[\frac{x^5}{5}\right]_0^1 = \frac{104}{5}$

or do it putting value of x and dx, y ranges from 0 to 4.
$x = \frac{\sqrt{y}}{2}$ and $dx = \frac{1}{2}\cdot\frac{1}{2\sqrt{y}}dy = \frac{1}{4\sqrt{y}}dy$.

$\int F\cdot dr = \int_0^4 (2x^2y\,dx + 3xy\,dy) = \int_0^4 \left[2\cdot\frac{y}{4}\cdot y\frac{dy}{4\sqrt{y}} + 3\cdot\frac{\sqrt{y}}{2}\cdot y\,dy\right]$

$= \int_0^4 \left[\frac{y^{3/2}}{8} + \frac{3}{2}y^{3/2}\right]dy = \frac{13}{8}\left[y^{5/2}\cdot\frac{2}{5}\right]_0^4 = \frac{13}{8}\times\frac{2}{5}\times 32 = \frac{104}{5}$ Same

EX② Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and c is the boundary of the square in the plane $z=0$ and bounded by lines $x=0,\ y=0,\ x=a,\ y=a$.



Sol^n: $\int_c \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$

Here $\vec{r} = \hat{i}x + \hat{j}y + 0\hat{k} \Rightarrow d\vec{r} = \hat{i}dx + \hat{j}dy$ ... ①

$\therefore \vec{F}\cdot d\vec{r} = (x^2\hat{i} + xy\hat{j})\cdot(\hat{i}dx + \hat{j}dy) = x^2dx + xy\,dy$

along OA, $y=0$, so $\vec{F}\cdot d\vec{r} = x^2dx$ $\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$. — ②

along AB, $x=a \Rightarrow dx=0$, so — ③

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int a y\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^2}{2}$$

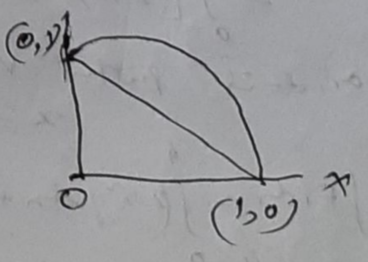
along BC $y=a,\ dy=0$, $\therefore \vec{F}\cdot d\vec{r} = x^2dx$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3} \quad ... ④$$

along CO, $x=0$, $\vec{F}\cdot d\vec{r} = 0$ $\therefore \int_{CO} \vec{F}\cdot d\vec{r} = 0$ ... ⑤

hence $\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + \frac{a^2}{2} - \frac{a^3}{3} + 0 = \frac{a^2}{2}$.

EX③ Evaluate the integral $\int_C (x^2dx + xy\,dy)$ taken ① the line segment from $(1,0)$ to $(0,1)$ ii) the quarter circle $x = \cos t,\ y = \sin t$ joining the same points.



Sol^n. The equ^n of the line joining the points $(1,0), (0,1)$ is $x+y=1 \Rightarrow$

$\Rightarrow y = 1-x \Rightarrow dx = -dy$

$$\int(x^2dx + xy\,dy) = \int_1^0 \{x^2 - x(1-x)\}dx$$

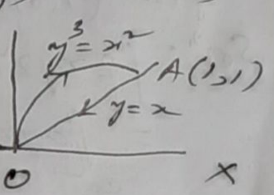
$$= \int_1^0 (2x^2 - x)\,dx = \left[\frac{2x^3}{3} - \frac{x^2}{2}\right]_0^1 = -\frac{1}{6}$$

(ii) $$\int_C (x^2dx + xy\,dy) = \int_0^{\pi/2} (-\cos^2 t\sin t + \cos^2 t\sin t)\,dt = 0.$$

or $$\int_1^0 \left[x^2dx + x\sqrt{1-x^2}\cdot\frac{-2x}{2\sqrt{1-x^2}}dx\right] = 0.$$ circle $x^2+y^2=1$.

Ex.3. Find the value of $\int\{(x+y^2)dx + (x^2-y)dy\}$ taken in the clock-wise sense along the ~~curve~~ closed curve C formed by $y^3 = x^2$ and the chord joining (0,0) to (1,1)

Sol^n: The curve C consist the arc $\overset{\frown}{OA}$ ($y^3 = x^2$) and the chord AO ($y = x$)



Now ~~for~~ along $\overset{\frown}{OA}$ through the curve $y^3 = x^2$

$$= \int^1 [(x+y^2)dx + (x^2-y)\,dy$$

$$= \int_0^1 [(x + x^{4/3})\cdot dx + (x^2 - x^{2/3})\frac{2}{3}\cdot x^{-1/3}dx$$

$$= \left[\frac{x^2}{2} + \frac{3x^{7/3}}{7} + \frac{2}{3}\left(x^{8/3}\cdot\frac{3}{8} - x^{4/3}\cdot\frac{3}{4}\right)\right]_0^1$$

$$= \frac{1}{2} + \frac{3}{7} + \frac{2}{3}\times -\frac{3}{8} = \frac{14+12-7}{28} = \frac{19}{28}$$

$\because x = y^{3/2}$

$dx = \frac{3}{2}y^{1/2}dy$

and $y = x^{2/3}$

$dy = \frac{2}{3}x^{-1/3}$

line integral alon $\overrightarrow{AO}$ through line $y = x$

$$= \int_0^1 (x+x^2)dx + (x^2-x)dx = \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{3} - \frac{x^2}{2}\right]_1^0$$

$$= -\frac{2}{3}$$

required line integral $\frac{19}{28} - \frac{2}{3} = \frac{57-56}{84} = \frac{1}{84}$.

Other-wise

P.T.O.

If we transfer the variable in y then for the curve $y^3 = x^2$
$\Rightarrow x = y^{3/2} \Rightarrow dx = \frac{3}{2} y^{\frac{1}{2}} dy$ and for $y = x \Rightarrow dy = dx$

1st Part:

$$\int_0^1 \left[\int (y^{3/2} + y^2) \cdot \frac{3}{2} y^{\frac{1}{2}} dy + (y^3 - y) dy = \int_0^1 \frac{3}{2}\left[(y^2 + y^{5/2}) + y^3 - y\right] dy\right.$$

$$= \frac{3}{2}\left[\frac{y^3}{3} + y^{7/2} \times \frac{2}{7} + \frac{y^4}{4} - \frac{y^2}{2}\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} + \frac{2}{7}\right) + \frac{1}{4} - \frac{1}{2}$$

$$= \frac{3}{2}\left(\frac{13}{21}\right) - \frac{1}{4} = \frac{13}{14} - \frac{1}{4} = \frac{26-7}{28} = \frac{19}{28}$$

Now line integral along st. line $y = x$ from A to O.

$$= \int_1^0 (y + y^2) dy + (y^2 - y) dy = \left[\frac{y^2}{2} + \frac{y^3}{3} - \frac{y^2}{2}\right]_1^0 = -\frac{2}{3}.$$

Hence required line integral $= \frac{19}{28} - \frac{2}{3} = \frac{57-56}{84} = \frac{1}{84}$.

EX ⑤ Determine whether the line integral

HW $\int 2xyz^2 dx + (x^2z^2 + z\cos yz) dy + (y\cos yz + 2x^2yz) dz$ is the independent of the path of integration. If so then evaluate it from $(1,0,1)$ to $(1, \frac{\pi}{2}, 1)$.

Sol^n: $\int 2xyz^2 dx + (x^2z^2 + z\cos yz) dy + (y\cos yz + 2x^2yz) dz$

$$= \int_C \left[2xyz^2 \hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + \cos yz)\hat{k}\right] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_C \vec{F} \cdot d\vec{r} \quad \text{where } \vec{F} = \vec{\nabla}\phi, \quad \phi \text{ is scalar potential.}$$

The integral is independent of path if $\vec{\nabla} \times \vec{F} = 0$. (irrotational)

Now
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix} = i(\quad) + j(\quad) + k(\quad) = 0$$

Hence the line integral is independent of Path

we know $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right) \cdot (i dx + j dy + k dz)$

$\therefore d\phi = (2xyz^2) dx + (x^2z^2 + z\cos yz) dy + (2x^2yz + y\cos yz) dz \equiv \vec{\nabla}\phi \cdot dr = \vec{F} \cdot dr$

$\phi = x^2yz^2 + 0 + z\left(+\frac{\sin yz}{z}\right) + \left(2x^2y\frac{z^2}{2}\right) + y\left(+\frac{\sin yz}{y}\right)$

$\phi = \int d(x^2yz^2 + \sin yz) = x^2yz^2 + \sin yz$

[integrating Partially omitting the same term

~~$d\phi$ $\phi = \int_{1,0,1}^{0,\frac{\pi}{2},1} x^2yz^2 + \sin yz \Big]_{1,0,1}^{0,\frac{\pi}{2},1} = [0+0]$~~ $\phi(B) - \phi(A)$

$$\phi = \left[x^2yz^2 + \sin yz\right]_{1,0,1}^{0,\frac{\pi}{2},1} = \sin\frac{\pi}{2} = 1$$

Or, $\left[\int d(x^2yz^2) + \int d(\sin yz)\right]_{1,0,1}^{0,\frac{\pi}{2},1} = \sin\frac{\pi}{2} = 1$

Ex. Find the work done in a moving particle of the force $\vec{F} = (2x - y + z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 3z)\hat{k}$ in an ellipse $z = 0$; $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Solⁿ Let $C: z = 0,\ \frac{x^2}{9} + \frac{y^2}{16} = 1$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = (-3\sin\theta\,\hat{i} + 4\cos\theta\,\hat{j})\,d\theta$

in Parametric form let $x = 3\cos\theta$, $y = 4\sin\theta$; where θ varies from 0 to 2π.

So,
$$\vec{F} \cdot d\vec{r} = \left\{(2 \times 3\cos\theta - 4\sin\theta + 0)\hat{i} + (3\cos\theta + 4\sin\theta)\hat{j}\right\} \cdot (-3\sin\theta\,\hat{i} + 4\cos\theta\,\hat{j})\,d\theta$$
$$= (-18\sin\theta\cos\theta + 12\sin^2\theta + 12\cos^2\theta + 16\sin\theta\cos\theta)\,d\theta.$$

Hence
$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (12 - \sin 2\theta)\,d\theta = \left[12\theta + \frac{\cos 2\theta}{2}\right]_0^{2\pi}$$
$$= 24\pi + 0 = 24\pi.$$

Ex. A vector field $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the integral over a circular path $x^2 + y^2 = a^2$, $z = 0$.

Solⁿ:
$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r} = \int_C \left[(\sin y)\hat{i} + x(1 + \cos y)\hat{j}\right] \cdot \left[dx\,\hat{i} + dy\,\hat{j}\right]; \quad z = 0,\ dz = 0.$$
$$= \int_C (\sin y\,dx + x\cos y\,dy + x\,dy) = \int_C d(x\sin y) + \int_C x\,dy$$

in the Parametric form the given path $x = a\cos\theta$, $y = a\sin\theta$; where θ ranges from 0 to 2π

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} d\left[a\cos\theta\sin(a\sin\theta)\right] + \int_0^{2\pi} a^2\cos^2\theta\,d\theta.$$
$$= \left[a\cos\theta\sin(a\sin\theta)\right]_0^{2\pi} + \int_0^{2\pi} \frac{1}{2}a^2(1 + \cos 2\theta)\,d\theta$$
$$= 0 \cancel{+ \frac{a^2}{2}\left[1 + \cos 2\theta\right]_0^{2\pi}} = \cancel{a^2}\ 0 + \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$
$$= \frac{a^2}{2} \cdot 2\pi = \pi a^2.$$

(EX): Prove that $\vec{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)\hat{k}$ is a Conservative field.
Find the scalar potential V such that $\vec{F} = \nabla V$.

Soln: If the given field F be conservative, then curl F = 0. Here

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

(is satisfied hence $\vec{F}$ is conservative field)

$$\hat{i}\left\{\frac{\partial}{\partial y}(3xz^2+2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right) + \hat{j}\left\{3z^2 - 3z^2\right) + k(2y\cos x - 2y\cos x) = 0$$

To find the scalar potential V of $\vec{F}$, we have

$$dV = \vec{F}\cdot d\vec{r} = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz = d(y^2\sin x) + d(xz^3) - 4dy + 2dz$$

$$= d(y^2\sin x + xz^3 - 4y + 2z)$$

$$\therefore V = y^2\sin x + xz^3 + 2z.$$

EX. Find the circulation of $\vec{F}$ round the curve where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the circle $x^2+y^2=1,\ z=0$

Soln: we have

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (y\hat{i} + z\hat{j} + x\hat{k})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \oint_C (y\,dx + z\,dy + x\,dz)$$ now for $z = 0$, $\Rightarrow dz = 0$

$$= \oint_C y\,dx$$ [let $x = \cos\theta$, $y = \sin\theta$. $dx = -\cos\theta$. for circle

$$= \oint \int_0^{2\pi} (\sin\theta)(-\sin\theta)\,d\theta = -4\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$-4\tfrac{1}{2}\int_0^{\pi/2} [1-\cos 2\theta)\,d\theta = -2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = -2\left(\frac{\pi}{2} - 0\right) = -\pi.$$

N.B. ① If the line integral of a vector field F depends on F and on the end points A, B but not on the nature of the path joining A and B, the we call the vector field F a conservative vector field. and $\nabla \times F = 0$.

② line integra $\oint F.dr$ around a closed curve c is called the circulation of the Vector field F round c. The field of F defined over a region R is then said to be irrotational if its circulation $\oint_c F.dr$ around all closed curves in R is zero.

A necessary and sufficient Condition that the continuous vector field $\vec{F}$ be irrotational in a simply connected region is that a single-valued scalar point function $\phi(x,y,z)$ exists for which grad $\phi = F$.

Ex. If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$ evaluate $\int_c \vec{F} \times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$

Soln. $$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = (-z\,dz - x\,dy)\hat{i} - (2y\,dz - x\,dx)\hat{j} + (2y\,dy + z\,dx)\hat{k}.$$

$$= [-2\cos t(-2\sin t)\,dt - \cos t(\cos t)\,dt]\hat{i} - [2\sin t(-2\sin t)\,dt - \cos t(-\sin t)\,dt]\hat{j} + [2\sin t(\cos t)\,dt - 2\cos t\sin t\,dt]\hat{k}$$

$$\therefore \int_C \vec{F} \times d\vec{r} = \int_0^{\pi/2} [(4\cos t\sin t - \cos^2 t)\hat{i} + (4\sin^2 t - \cos t\sin t)\hat{j}]$$

$$= \int_0^{\pi/2} \left\{2\sin 2t - \frac{\cos 2t + 1}{2}\right\}\hat{i}\,dt + \int_0^{\pi/2} \left\{2(1 - \cos 2t) - \frac{1}{2}\sin 2t\right\}\hat{j}$$

do it

$$= \left(2 - \frac{\pi}{4}\right)i + \left(\pi - \frac{1}{2}\right)$$