Semester-II

Course Type- Core-3

Course Title - C3T: Real Analysis

Topic- Monotone sequence

References: S.K. Mapa Book.

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Date- 05.04.2020

1: Let $u_n = 2^n, n \geq 1$. Then $u_{n+1} > u_n$ for all $n \in \mathbb{N}$. Therefore the sequence $\{u_n\}$ is a monotone increasing sequence. It is also strictly monotone sequence.

$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+(n-1)} + \frac{1}{n+n}$

$u_{n+1} = \frac{1}{n+1+1} + \frac{1}{n+1+2} + \cdots + \frac{1}{n+1+(n+1-1)} + \frac{1}{n+1+n+1}$

$= \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}$

2: Let $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots\ldots + \frac{1}{2n}$, $n \geq 1$.

Then $u_{n+1} - u_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} > 0$ for all $n \in \mathbb{N}$. $\Rightarrow$

Therefore the sequence $\{u_n\}$ is a monotone increasing sequence. It is also strictly monotone sequence.

3: The sequence $\{(-2)^n\}$ is neither a monotone increasing sequence nor a monotone decreasing sequence. Therefore it is not a monotone sequence.

Theorem: A monotone increasing sequence, if bounded above, is convergent and converges to its least upper bound. **VU'1998, 03, 05, CU'2000,** VU-'11

Proof: Let $\{u_n\}$ be a monotone increasing sequence bounded above and let M be its least upper bound. Then

(i) $u_n \leq M$ for all $n \in \mathbb{N}$ and

(ii) For a pre-assigned positive ε, there exists a natural number k such that $u_k > M - \varepsilon$.

Since $\{u_n\}$ is a monotone increasing sequence, $M - \varepsilon < u_k \leq u_{k+1} \leq u_{k+2} \leq \ldots\ldots\ldots\ldots \leq M$

That is, $M - \varepsilon < u_n < M + \varepsilon$ for all $n \geq k$.

This shows that the sequence $\{u_n\}$ is convergent and $\lim_{n\to\infty} u_n = M$.

Theorem: A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

(H.W.)

Ex 25: Show that a necessary and sufficient condition for the convergence of a monotone increasing sequence is that it is bounded above. **CU'2002, 04, 07**

(H.W.)

Theorem: A monotone increasing sequence that is unbounded above diverges to ∞.

Proof: Let $\{u_n\}$ be a monotone increasing sequence, not bounded above. Since the sequence in unbounded above, for a pre-assigned positive number G, however large, there exists a natural number k such that $u_k > G$.

Since the sequence $\{u_n\}$ is monotone increasing, $G < u_k \leq u_{k+1} \leq u_{k+2} \leq \ldots\ldots\ldots\ldots$

That is, $u_n > G$ for all $n \geq k$.

This proves that the sequence $\{u_n\}$ diverges to ∞.

Theorem: A monotone decreasing sequence that is unbounded below diverges to $-\infty$.

Theorem (Cantor's theorem on nested intervals):

Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals such that

(i) $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ For all $n \in \mathbb{N}$, and

(ii) $\lim_{n\to\infty} \delta_n = 0$ Where $\delta_n = b_n - a_n =$ length of $[a_n, b_n]$.

Then $\bigcap_{n=1}^{\infty}[a_n, b_n]$ contains precisely one point

Ex 26: Prove that the sequence $\left\{n^{\frac{1}{13}}\right\}$ diverge to ∞. **VU'2005**

Hints: The sequence $\left\{n^{\frac{1}{13}}\right\}$ is a monotone increasing sequence unbounded above. Therefore diverge to ∞.

Another method: Let G be a +ve real number. Then $n^{\frac{1}{13}} > G$ if $n > G^{13}$

Let $k = \left[G^{13}\right]+1$ then k is a natural number and $n^{\frac{1}{13}} > G$ holds for all $n \ge k$

$\Rightarrow \left\{n^{\frac{1}{13}}\right\}$ diverge to ∞

Ex 27: Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ lies between 2 and 3.

Let $u_n = \left(1+\frac{1}{n}\right)^n$. Then $u_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$

Let us consider $n+1$ positive numbers $1+\frac{1}{n}, 1+\frac{1}{n}, \ldots\ldots\ldots\ldots, 1+\frac{1}{n}$ (n times) and 1.

Applying $A.M. > G.M.$, we have $\dfrac{n\left(1+\frac{1}{n}\right)+1}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

Or, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$

i.e., $u_{n+1} > u_n$ for all $n \in \mathbb{N}$

This shows that the sequence $\{u_n\}$ is a monotone increasing sequence.

$$\text{Now } u_n = 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\ldots\ldots\ldots\ldots+\frac{n(n-1)\ldots\ldots\ldots2.1}{n!}\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots\ldots\ldots\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\frac{2}{n}.\frac{1}{n}$$

$$<1+1+\frac{1}{2!}+\ldots\ldots\ldots\ldots+\frac{1}{n!} \quad \text{for all } n \ge 2.$$

We have $n! > 2^{n-1}$ for all $n > 2$.

$$1+1+\frac{1}{2!}+\ldots\ldots\ldots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\ldots\ldots+\frac{1}{2^{n-1}}, \text{ for } n > 2$$

$$\text{Also } 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\ldots+\frac{1}{2^{n-1}} = 1+2\left[1-\left(\frac{1}{2}\right)^n\right] < 3 \text{ for all } n \in \mathbb{N}$$

It follows that $u_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence bounded above, is convergent.

The limit of the sequence is denoted by e. That is, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, where $2 < e < 3$.

Ex 28: Show that the sequence $\{x_n\}$ where $x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots\ldots\ldots+\frac{1}{n!}$ is convergent and $\lim_{n\to\infty} x_n = e$.

$$x_{n+1} - x_n = \frac{1}{(n+1)!} > 0 \quad \text{For all } n \geq 1$$

So $x_{n+1} > x_n$ for all $n \geq 1$

This shows that the sequence $\{x_n\}$ is a monotone increasing sequence.

$$x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots\ldots\ldots+\frac{1}{n!}$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\ldots\ldots+\frac{1}{2^{n-1}}, \text{ for } n \geq 3, \text{ since } n! > 2^{n-1} \text{ for all } n \geq 3$$

Again $1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\ldots\ldots+\frac{1}{2^{n-1}} = 1+2\left[1-\left(\frac{1}{2}\right)^n\right] < 3$ for all $n \in \mathbb{N}$

It follows that $x_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{x_n\}$ is bounded above.

Thus the sequence $\{x_n\}$ being a monotone increasing sequence bounded above, is convergent.

Let $u_n = \left(1+\frac{1}{n}\right)^n$.

Then $u_n = 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots\ldots\ldots\ldots+\frac{n(n-1)\ldots\ldots\ldots\ldots2.1}{n!}\cdot\frac{1}{n^n}$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots\ldots\ldots\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\frac{2}{n}\cdot\frac{1}{n}$$

$$< 1+1+\frac{1}{2!}+\ldots\ldots\ldots\ldots+\frac{1}{n!} \quad \text{for all } n \geq 2.$$

Therefore $\lim_{n\to\infty} u_n \leq \lim_{n\to\infty} x_n$ (since both the limits exist)

Or, $e \leq \lim_{n\to\infty} x_n$ .... (A)

Let us choose a natural number m. Then for each $n > m$,

$$u_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots\ldots\ldots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{m-1}{n}\right)+\ldots\ldots\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\frac{2}{n}\cdot\frac{1}{n}$$

$$> 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots\ldots\ldots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\left(1-\frac{m-1}{n}\right).$$

Keeping m fixed, let $n \to \infty$. Then $\lim_{n\to\infty} u_n \geq 1+1+\frac{1}{2!}+\ldots\ldots\ldots+\frac{1}{m!}$

Or, $e \geq 1+1+\frac{1}{2!}+\ldots\ldots\ldots+\frac{1}{m!}$

Or, $x_m \leq e$.

Proceeding to limit as $m \to \infty$, $\lim_{m\to\infty} x_m \leq e$ .... (B)

From (A) and (B), $\lim_{n\to\infty} x_n = e$

Ex 29: Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a monotone decreasing sequence with limit e.

✍ Let $v_n = \left(1+\frac{1}{n}\right)^{n+1}$

Let us consider $n+2$ positive numbers

$1-\frac{1}{n+1}$, $1-\frac{1}{n+1}$,............., $1-\frac{1}{n+1}$ [$(n+1)$ times] and 1.

Applying $A.M. > G.M.$, we have $\frac{(n+1)\left(1-\frac{1}{n+1}\right)+1}{n+2} > \left(1-\frac{1}{n+1}\right)^{\frac{n+1}{n+2}}$

Or, $\left(\frac{n+1}{n+2}\right)^{n+2} > \left(\frac{n}{n+1}\right)^{n+1}$

Or, $\left(\frac{n+1}{n}\right)^{n+1} > \left(\frac{n+2}{n+1}\right)^{n+2}$

Or, $\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$

i.e., $v_n > v_{n+1}$ for all $n \in \mathbb{N}$

This shows that the sequence $\{v_n\}$ is a monotone decreasing sequence.

Again $v_n = 1+\frac{n+1}{n}+\frac{(n+1)n}{2!}\cdot\frac{1}{n^2}+\dots\dots\dots+\frac{1}{n^{n+1}} > 1$ for all $n \in \mathbb{N}$

This shows that the sequence $\{v_n\}$ is bounded below.

Hence the sequence $\{v_n\}$ is convergent.

Let $u_n = \left(1+\frac{1}{n}\right)^n$. Then $v_n - u_n = \left(1+\frac{1}{n}\right)^n\cdot\frac{1}{n}$ and $\lim_{n\to\infty}(v_n - u_n) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{n} = 0$.

This implies that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$, since both the limits exist.

As $\lim_{n\to\infty} u_n = e$, it follows that $\lim_{n\to\infty} v_n = e$

Ex 30: Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for all $n \geq 1$ converges to 2. **VU'2006‖CU'1999**

✍ The sequence is $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\dots\dots\dots\right\}$

$u_{n+1}^2 - u_n^2 = 2(u_n - u_{n-1})$

Or, $(u_{n+1}+u_n)(u_{n+1}-u_n) = 2(u_n - u_{n-1})$.

Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently $u_3 > u_2, u_4 > u_3, \dots\dots\dots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $2u_n = u_{n+1}^2 > u_n^2$ for all $n \in \mathbb{N}$

That is, $u_n^2 - 2u_n < 0$ for all $n \in \mathbb{N}$.

Or, $u_n(u_n-2)<0$ for all $n \in \mathbb{N}$

But $u_n>0$. Therefore $u_n<2$ for all $n \in \mathbb{N}$

This shows that the sequence $\{u_n\}$ is bounded above and therefore it is convergent.

Let $\lim\limits_{n\to\infty} u_n = l$.

By definition, $u_{n+1}^2 = 2u_n$ for all $n \in \mathbb{N}$

Taking limit as $n \to \infty$, we have $l^2 = 2l$. Therefore l is either 0 or 2.

But l cannot be 0 since the sequence $\{u_n\}$ is monotone increasing and $u_1 = \sqrt{2} > 1$.

Therefore $l = 2$. That is, the sequence converges to 2.

Ex 31: Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7+u_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

The sequence is $\left\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\ldots\ldots\ldots\right\}$.

$u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$

Or, $(u_{n+1}+u_n)(u_{n+1}-u_n) = u_n - u_{n-1}$.

Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently, $u_3 > u_2$, $u_4 > u_3, \ldots\ldots\ldots\ldots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $u_n^2 < u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$

Or, $u_n^2 - u_n - 7 < 0$

Or, $(u_n - \alpha)(u_n - \beta) < 0$ where α, β are the roots of the equation $x^2 - x - 7 = 0$. One of the roots is negative and the other is positive.

Let $\alpha < 0$.

Since $u_n > 0$ for all $n \in \mathbb{N}$, $u_n - \alpha > 0$. Consequently, $u_n < \beta$ for all $n \in \mathbb{N}$

This proves that the sequence $\{u_n\}$ is bounded above and therefore the sequence $\{u_n\}$ is convergent.

Let $\lim\limits_{n\to\infty} u_n = l$.

By definition, $u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$

Taking limit as $n \to \infty$, we have $l^2 = 7 + l$.

Therefore $(l-\alpha)(l-\beta) = 0$

But $l \neq \alpha$, since each element of the sequence is positive and $\alpha < 0$.

Therefore $l = \beta$. That is, the sequence converges to the positive roots of the equation $x^2 - x - 7 = 0$.

Ex 32: Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6+u_n}$ for $n \geq 1$ converges to 3. **CU'2004**

(H.W.)

Ex 33: If $x_{n+1} = \sqrt{k + x_n}$ where x_n and k are +ve. Show that $\{x_n\}$ is increasing or decreasing according as x is less then or greater then the +ve root of $x^2 - x - k$ and has and has in either case this root as its limit. **VU'1999**

(H.W.)

Ex 34: Show that the sequence $\{u_n\}$ where $u_n = 1+\frac{1}{2}+\frac{1}{3}+..........+\frac{1}{n}-\log n$ is convergent.

CU'2007

$$u_{n+1}-u_n = \frac{1}{n+1}-\log(n+1)+\log n = \frac{1}{n+1}-\log\left(1+\frac{1}{n}\right).$$

As the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a strictly monotone decreasing sequence converging to e,

$\left(1+\frac{1}{n}\right)^{n+1} > e$ for all $n \in \mathbb{N}$.

Therefore $\log\left(1+\frac{1}{n}\right) > \frac{1}{n+1}$ for all $n \in \mathbb{N}$

Or, $u_{n+1} < u_n$ for all $n \in \mathbb{N}$

This shows that the sequence $\{u_n\}$ is a strictly monotone decreasing sequence.

Again $\frac{1}{n} > \log\frac{n+1}{n} = \log(n+1)-\log n$

Therefore $1 > \log 2 - \log 1, \frac{1}{2} > \log 3 - \log 2, ..., \frac{1}{n} > \log(n+1)-\log n$

So we have $1+\frac{1}{2}+.........+\frac{1}{n} > \log(n+1) > \log n.$

Hence $u_n > 0$ for all $n \in \mathbb{N}$

Therefore $\{u_n\}$ is a monotone decreasing sequence bounded below

Hence the sequence $\{u_n\}$ is convergent.

Note: $1+\frac{1}{2}+.........+\frac{1}{n}-\log n$ is denoted by γ_n. Then the sequence $\{\gamma_n\}$ is convergent and the limit of this sequence is denoted by γ. γ is called an Euler's constant.

Ex 35: Two sequences $\{x_n\}$, $\{y_n\}$ are defined by $x_{n+1} = \frac{1}{2}(x_n + y_n)$, $y_{n+1} = \sqrt{x_n y_n}$ for $n \geq 1$ and $x_1 > 0$, $y_1 > 0$. Prove that both the sequences converge to a common limit.

Case1: Let $x_1 \neq y_1$.

$x_2 = \frac{1}{2}(x_1 + y_1) > \sqrt{x_1 y_1} = y_2$.

Let us assume that $x_k > y_k$.

Then $x_{k+1} = \frac{1}{2}(x_k + y_k) > \sqrt{x_k y_k} = y_{k+1}$

$x_k > y_k$ implies $x_{k+1} > y_{k+1}$ and $x_2 > y_2$.

By the principle of induction, $x_n > y_n$ for all $n \geq 2$

$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n$ for all $n \geq 2$.

$y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n . y_n} = y_n$ for all $n \geq 2$.

So we have $y_2 < y_3 < y_4 < \ldots\ldots\ldots\ldots < x_4 < x_3 < x_2$.

Therefore the sequence $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and the sequence $\{y_n\}_{n=2}^{\infty}$ is a monotone increasing sequence bounded above. Hence both the sequences are convergent.

Let $\lim\limits_{n\to\infty} x_n = l$, $\lim\limits_{n\to\infty} y_n = m$.

$x_{n+1} = \frac{1}{2}(x_n + y_n)$ for all $n \in \mathbb{N}$.

Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l+m)$. i.e., $l = m$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ converge to a common limit.

__Case2:__ Let $x_1 = y_1$.

In this case $x_n = y_n = x_1$ for all $n \in \mathbb{N}$

Therefore $\{x_n\}$ and $\{y_n\}$ both converge to the same limit x_1.

__Ex 36:__ If $u_1 > 0$ and $u_{n+1} = \frac{1}{2}\left(u_n + \frac{9}{u_n}\right)$ for $n \geq 1$, prove that the sequence $\{u_n\}$ converges to 3.

$u_n^2 - 2u_n u_{n+1} + 9 = 0$. This is a quadratic equation in u_n having real roots.

Therefore $4u_{n+1}^2 - 36 \geq 0$

This implies $u_{n+1} \geq 3$ for all $n \geq 1$, since $u_{n+1} > 0$ for all $n \geq 1$.

$$u_n - u_{n+1} = u_n - \frac{1}{2}\left(u_n + \frac{9}{u_n}\right) = \frac{1}{2}\left(\frac{u_n^2 - 9}{u_n}\right) \geq 0 \text{ for all } n \geq 2.$$

Therefore $u_{n+1} \leq u_n$ for all $n \geq 2$

This shows that the sequence $\{u_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and hence the sequence $\{u_n\}$ is convergent.

Let $\lim\limits_{n\to\infty} u_n = l$.

$u_{n+1} = \frac{1}{2}\left(u_n + \frac{9}{u_n}\right)$ for $n \geq 1$. Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}\left(l + \frac{9}{l}\right)$. This gives $l = 3$, since $l > 0$.

__Ex 37:__ Prove that the sequence $\{x_n\}$ defined by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$ for $n \geq 1$, $\alpha > 0$ and $x_1 \geq \sqrt{\alpha}$, is a convergent sequence. Find the limit.

$x_n^2 - 2x_n x_{n+1} + \alpha = 0$. This is a quadratic equation in α_n having real roots.

Therefore $4x_{n+1}^2 - 4\alpha \geq 0$

This implies $x_{n+1} \geq \alpha$ for all $n \geq 1$, since $x_{n+1} > 0$ for all $n \geq 1$.

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\frac{x_n^2 - \alpha}{x_n}\right) \geq 0 \text{ for all } n \geq 1.$$

Therefore $x_{n+1} \le x_n$ for all $n \ge 1$

This shows that the sequence $\{x_n\}$ is a monotone decreasing sequence bounded below and hence the sequence $\{x_n\}$ is convergent.

Let $\lim_{n\to\infty} x_n = l$.

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$ for $n \ge 1$. Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}\left(l + \frac{\alpha}{l}\right)$. This gives $l = \pm\sqrt{\alpha}$.

But l cannot be $-\sqrt{\alpha}$, since $\{x_n\}$ is a sequence of positive real numbers.

Therefore $l = \sqrt{\alpha}$

Ex 38: Let $\{u_n\}$ be a sequence defined by $u_{n+1} = \frac{1}{k}\left(u_n + \frac{k}{u_n}\right)$, $k > 1$ and $a_1 > 0$, show that $\{u_n\}$ converges to $\sqrt{\frac{k}{k-1}}$.

✍ (H.W.)

Ex 39: If $s_1 > 0$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{4}{s_n}\right)$ for $n \ge 1$, prove that the sequence $\{s_n\}$ is monotone decreasing sequence bounded below and $\lim_{n\to\infty} s_n = 2$.

✍ (H.W.)

Ex 40: If $0 < u_1 < 1$ and $u_{n+1} = 1 - \sqrt{1 - u_n}$ for $n \ge 1$, prove that (i) the sequence $\{u_n\}$ converges to 0 and (ii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2}$.

✍ (i) $1 - u_{n+1} = (1 - u_n)^{\frac{1}{2}} = (1 - u_{n-1})^{\frac{1}{2^2}} = \dots\dots\dots = (1 - u_1)^{\frac{1}{2^n}}$ for all $n \in \mathbb{N}$.

Since $0 < u_1 < 1$, therefore $0 < 1 - u_1 < 1$

Thus $\lim_{n\to\infty}(1 - u_{n+1}) = \lim_{n\to\infty}(1 - u_1)^{\frac{1}{2^n}} = 1$, since $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Therefore $1 - \lim_{n\to\infty} u_n = 1$ or, $\lim_{n\to\infty} u_n = 0$

(ii) Since $u_{n+1} = 1 - \sqrt{1 - u_n}$ for all $n \in \mathbb{N}$.

Therefore $\frac{u_{n+1}}{u_n} = \frac{1}{1 + \sqrt{1 - u_n}}$ for all $n \in \mathbb{N}$

Thus we have $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{1 + \sqrt{1 - u_n}} = \frac{1}{2}$, since $\lim_{n\to\infty} u_n = 0$.

Ex 41: A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \sqrt{6 + u_n}$ for $n \ge 1$. Show that (i) the sequence $\{u_n\}$ is monotone increasing if $0 < u_1 < 3$; (ii) the sequence $\{u_n\}$ is monotone decreasing if $u_1 > 3$. Find $\lim_{n\to\infty} u_n$.

Since $u_1 > 0$, therefore $u_n > 0$ for all $n \in \mathbb{N}$,

$u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$

Or, $(u_{n+1} + u_n)(u_{n+1} - u_n) = u_n - u_{n-1}$.

Since $u_n > 0$ for all $n \in \mathbb{N}$, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$ .... (1)

(i) If $0 < u_1 < 3$ then $(u_1 + 2)(u_1 - 3) < 0$ i.e., $u_1^2 < u_1 + 6 = u_2^2$, i.e., $u_1 < u_2$, since $u_1, u_2 > 0$.

Consequently from (1), $u_2 < u_3$, $u_3 < u_4$,..................

Therefore $\{u_n\}$ is a monotone increasing sequence if $0 < u_1 < 3$.

(ii) If $u_1 > 3$ then $(u_1 + 2)(u_1 - 3) > 0$ i.e., $u_1^2 > u_1 + 6 = u_2^2$, i.e., $u_1 > u_2$, since $u_1, u_2 > 0$.

Consequently from (1), $u_2 > u_3$, $u_3 > u_4$,..................

Therefore $\{u_n\}$ is a monotone decreasing sequence if $u_1 > 3$.

2nd Part: If $0 < u_1 < 3$ then $\{u_n\}$ is monotone increasing and $u_{n+1}^2 > u_n^2$ for all $n \in \mathbb{N}$.

i.e., $u_n^2 - u_n - 6 < 0$ or, $(u_n + 2)(u_n - 3) < 0$ or $u_n < 3$ for all $n \in \mathbb{N}$, since $u_n > 0$.

Thus if $0 < u_1 < 3$ then $\{u_n\}$ is monotone increasing sequence bounded above and is convergent.

If $u_1 > 3$ then $\{u_n\}$ is monotone decreasing and $u_{n+1}^2 < u_n^2$ for all $n \in \mathbb{N}$

i.e., $u_n^2 - u_n - 6 > 0$ or, $(u_n + 2)(u_n - 3) > 0$ or $u_n > 3$ for all $n \in \mathbb{N}$, since $u_n > 0$.

Thus if $u_1 > 3$ then $\{u_n\}$ is monotone decreasing sequence bounded below and is convergent.

Let $\lim_{n\to\infty} u_n = l$.

By definition, $u_{n+1}^2 = 6 + u_n$ for all $n \in \mathbb{N}$

Taking limit as $n \to \infty$ we have, $l^2 = 6 + l$ or, $l^2 - l - 6 = 0$ or, $(l+2)(l-3) = 0$

Either $l = -2$ or $l = 3$.

But l cannot be -2, since $u_n > 0$ for all $n \in \mathbb{N}$.

Therefore $l = 3$, i.e., $\lim_{n\to\infty} u_n = 3$

Ex 42: Prove that the sequence $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = \frac{1}{2}(x_n + y_n)$, $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1$, $x_1 > 0$, $y_1 > 0$ converges to a common limit l where $l^2 = x_1 y_1$.

Case1: $x_1 \neq y_1$

$$x_2 = \frac{1}{2}(x_1 + y_1) > \frac{2}{\frac{1}{x_1} + \frac{1}{y_1}} = y_2.$$

Let us assume that $x_k > y_k$.

$$\text{Then } x_{k+1} = \frac{1}{2}(x_k + y_k) > \frac{2}{\frac{1}{x_k} + \frac{1}{y_k}} = y_{k+1}$$

$x_k > y_k \Rightarrow x_{k+1} > y_{k+1}$ and $x_2 > y_2$.

By the principle of induction, $x_n > y_n$ for all $n \geq 2$

$$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n \quad \text{for all } n \geq 2.$$

$$y_{n+1} = \frac{2}{\frac{1}{x_n}+\frac{1}{y_n}} > \frac{2}{\frac{1}{y_n}+\frac{1}{y_n}} = y_n \quad \text{for all } n \geq 2.$$

So we have $y_2 < y_3 < y_4 < \ldots\ldots\ldots\ldots < x_4 < x_3 < x_2$.

Therefore the sequence $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and the sequence $\{y_n\}_{n=2}^{\infty}$ is a monotone increasing sequence bounded above. Hence both the sequences are convergent.

Let $\lim_{n\to\infty} x_n = l$, $\lim_{n\to\infty} y_n = m$.

$$x_{n+1} = \frac{1}{2}(x_n + y_n) \quad \text{for all } n \in \mathbb{N}.$$

Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l+m)$. i.e., $l = m$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ converge to a common limit.

Case2: Let $x_1 = y_1$.

In this case $x_n = y_n = x_1$ for all $n \in \mathbb{N}$

Therefore $\{x_n\}$ and $\{y_n\}$ both converge to the same limit x_1.

Now, $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n} = \frac{x_n + y_n}{x_n y_n} = \frac{2x_{n+1}}{x_n y_n}$ i.e., $x_{n+1}y_{n+1} = x_n y_n$

Thus we have, $x_{n+1}y_{n+1} = x_n y_n = x_{n-1}y_{n-1} = \ldots\ldots\ldots\ldots = x_2 y_2 = x_1 y_1$ for all $n \in \mathbb{N}$.

2nd Part: Let $\lim_{n\to\infty} x_n = l$, $\lim_{n\to\infty} y_n = l$. Then we have from above $l^2 = x_1 y_1$.

Ex 43: Let S be a non-empty subset of $\mathbb{R}$ having a limit point l. Show that there exists a sequence $\{u_n\}$ of distinct elements of S such that $\lim_{n\to\infty} u_n = l$.

✍ Let us choose $\varepsilon_1 > 0$. Since l is a limit point of S, there exists a point $u_1 (\neq l)$ of S such that $|u_1 - l| < \varepsilon_1$.

Let $\varepsilon_2 = |u_1 - l|$. Then $\varepsilon_2 > 0$ and $\varepsilon_1 > \varepsilon_2$.

Since l is a limit point of S, there exists a point $u_2 (\neq l)$ of S such that $|u_2 - l| < \varepsilon_2$.

Therefore $u_1 \neq u_2$

Let $\varepsilon_3 = |u_2 - l|$. Then $\varepsilon_3 > 0$ and $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.

Since l is a limit point of S, there exists a point $u_3 (\neq l)$ of S such that $|u_3 - l| < \varepsilon_3$.

Therefore $u_2 \neq u_3$

Proceeding in this way we get a sequence $\{u_n\}$ of distinct elements of S.

Let us choose $\varepsilon > 0$.

Since $\{\varepsilon_n\}$ is a decreasing sequence of positive real numbers, there exists a natural number k such that $\varepsilon_k < \varepsilon$ i.e., $\varepsilon_n < \varepsilon$ for all $n \geq k$.

So we have, $|u_n - l| < \varepsilon_n < \varepsilon$ for all $n \geq k$.

This implies that $\lim_{n\to\infty} u_n = l$.

Ex 44: Let S be an infinite subset of $\mathbb{R}$ that is bounded above and let $\sup S \notin S$. Show that there exists a monotone increasing sequence $\{u_n\}$ with $u_n \in S$, such that $\lim_{n\to\infty} u_n = \sup S$.

Let $\sup S = l$

Since $l \notin S$, $x < l$ for all $x \in S$.

Let us choose $\varepsilon_1 > 0$. Since $l = \sup S$, there exists a point u_1 of S such that $u_1 > l - \varepsilon_1$.

Let $\varepsilon_2 = l - u_1$. Then $\varepsilon_2 > 0$ and $\varepsilon_1 > \varepsilon_2$.

Since $l = \sup S$, there exists a point u_2 of S such that $u_2 > l - \varepsilon_2$.

Therefore $u_1 < u_2$, since $u_1 = l - \varepsilon_2$

Let $\varepsilon_3 = l - u_2$. Then $\varepsilon_3 > 0$ and $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$.

Since $l = \sup S$, there exists a point u_3 of S such that $u_3 > l - \varepsilon_3$.

Therefore $u_2 < u_3$, since $u_2 = l - \varepsilon_3$

Proceeding in this way we get a monotone increasing sequence $\{u_n\}$ where $u_n \in S$ for all $n \in \mathbb{N}$.

Let us choose $\varepsilon > 0$.

Since $\{\varepsilon_n\}$ is a decreasing sequence of positive real numbers, there exists a natural number k such that $\varepsilon_k < \varepsilon$ i.e., $\varepsilon_n < \varepsilon$ for all $n \geq k$.

So we have, $|u_n - l| < \varepsilon_n < \varepsilon$ for all $n \geq k$.

This implies that $\lim_{n\to\infty} u_n = l = \sup S$.

Ex 45: Give an example of a sequence of rational numbers that converges to an irrational number.

The sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a sequence of rational number which converges to e, an irrational number.

Ex 46: Give an example of a sequence of irrational numbers that converges to a rational number.

The sequence $\{u_n\}$ where $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{u_n}$ for all $n > 1$ is a sequence of irrational number which converges to 2, which is a rational number

Ex 47: Give an example of divergent sequences $\{x_n\}$ and $\{y_n\}$ such that the sequence $\{x_n + y_n\}$ is convergent.

Clearly $\{x_n\} = \{2^n\}$ is a divergent sequence and $\{y_n\} = \{-2^n\}$ is a divergent sequence but $\{x_n + y_n\} = \{2^n - 2^n\} = \{0,0,0,......\}$ is convergent.

Ex 48: Let $\{u_n\},\{v_n\}$ be two real sequences with $\lim_{n\to\infty} u_n = l$, $\lim_{n\to\infty} v_n = m$. If $x_n = \max\{u_n, v_n\}$, $y_n = \min\{u_n, v_n\}$ prove that the sequence $\{x_n\}$ converges to $\max\{l, m\}$ and the sequence $\{y_n\}$ converges to $\min\{l, m\}$.

Since $x_n = \max\{u_n, v_n\} \Rightarrow x_n = \{u_n + v_n + |u_n - v_n|\}\frac{1}{2}$

Since $\{u_n\}$ and $\{v_n\}$ are convergent therefore $\{u_n + v_n\}$ and $\{u_n - v_n\}$ are convergent.

Since $\{u_n - v_n\}$ is convergent $\Rightarrow \{|u_n - v_n|\}$ is convergent.

Since $\{u_n + v_n\}$ and $\{|u_n - v_n|\}$ are convergent therefore $\{u_n + v_n + |u_n - v_n|\}$ i.e. $\{a_n\}$ is convergent.

Again since $\lim\limits_{n\to\infty} u_n = l$ and $\lim\limits_{n\to\infty} v_n = m$ therefore $\lim\limits_{n\to\infty}(u_n + v_n) = l + m$ and $\lim\limits_{n\to\infty}|u_n - v_n| = |l - m|$

Therefore $\lim\limits_{n\to\infty} a_n = l + m + |l - m| = \max\{l, m\}$

Ex 49: Prove that $\lim\limits_{n\to\infty}(n+1)^{\frac{1}{n}} = 1$ and $\lim\limits_{n\to\infty} n^{\frac{1}{n+1}} = 1$.

$$\lim_{n\to\infty}(n+1)^{\frac{1}{n}} = \lim_{n\to\infty}\left[(n+1)^{\frac{1}{n+1}}\right]^{\frac{n+1}{n}} = \left[\lim_{n\to\infty}(n+1)^{\frac{1}{n+1}}\right]^{\lim\limits_{n\to\infty}\frac{n+1}{n}} = 1$$

$$\lim_{n\to\infty} n^{\frac{1}{n+1}} = \lim_{n\to\infty}\left[n^{\frac{1}{n}}\right]^{\frac{n}{n+1}} = \left[\lim_{n\to\infty} n^{\frac{1}{n}}\right]^{\lim\limits_{n\to\infty}\frac{n}{n+1}} = 1$$

Ex 50: A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \dfrac{3(1+u_n)}{5+u_n}$ for $n \geq 1$. Prove that (i) the sequence $\{u_n\}$ is a decreasing sequence if $u_1 > 1$; (ii) the sequence $\{u_n\}$ is an increasing sequence if $0 < u_1 < 1$; (iii) $\lim\limits_{n\to\infty} u_n = 1$ in both cases.

(i) $\dfrac{u_{n+1}}{u_n} = \dfrac{3(1+u_n)}{5+u_n}\cdot\dfrac{5+u_{n-1}}{3(1+u_{n-1})} = \dfrac{(1+u_n)(5+u_{n-1})}{(5+u_n)(1+u_{n-1})}$

Thus $u_{n+1} < u_n$ if $(1+u_n)(5+u_{n-1}) < (5+u_n)(1+u_{n-1})$

I.e. if $5 + u_{n-1} + 5u_n + u_n u_{n-1} < 5 + u_n + u_n u_{n-1} + 5u_{n-1}$

I.e. if $u_n < u_{n-1}$

Now $\dfrac{u_2}{u_1} = \dfrac{3(1+u_1)}{(5+u_1)u_1} = \dfrac{3(1+u_1)}{5u_1 + u_1^2}$

Clearly $u_2 < u_1$ if $3 + 3u_1 < 5u_1 + u_1^2$ i.e. if $u_1^2 + 2u_1 - 3 > 0$

But $u_1^2 + 2u_1 - 3 = (u_1+3)(u_1-1) > 0$ if $u_1 > 1$

Therefore when $u_1 > 1$ then $u_2 < u_1$

Also $u_{n+1} < u_n$ if $u_n < u_{n-1}$

Thus by principle of induction $u_{n+1} < u_n$ for all $n \in \mathbb{N}$

$\therefore \{u_n\}$ is a monotone decreasing sequence if $u_1 > 1$

(ii) Again $u_{n+1} > u_n$ if $u_n > u_{n-1}$

Clearly $u_2 > u_1$ if $u_1^2 + 2u_1 - 3 < 0$

But $u_1^2 + 2u_1 - 3 = (u_1+3)(u_1-1) < 0$ if $0 < u_1 < 1$

Therefore when $0 < u_1 < 1$ then $u_2 > u_1$

Also $u_{n+1} > u_n$ if $u_n > u_{n-1}$

Thus by principle of induction $u_{n+1} > u_n$ for all $n \in \mathbb{N}$

$\therefore \{u_n\}$ is a monotone increasing sequence if $0 < u_1 < 1$

(iii) Let $\lim_{n\to\infty} u_n = l$ then $\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \frac{3(1+u_n)}{5+u_n} = \frac{3\left(1+\lim_{n\to\infty} u_n\right)}{5+\lim_{n\to\infty} u_n}$

$\Rightarrow l = \frac{3(1+l)}{5+l} \Rightarrow l^2 + 2l - 3 = 0 \Rightarrow (l+3)(l-1) = 0$

Since $\{u_n\}$ is a sequence of +ve real numbers therefore $l \neq -3$.

Thus $l = 1$ i.e. $\lim_{n\to\infty} u_n = 1$

(23) $0 < u_1 < u_2$, $u_{n+2} = \sqrt{u_{n+1} u_n}$.

$\therefore u_n = \sqrt{u_{n-1} u_{n-2}}$

$u_{n-1} = \sqrt{u_{n-2} u_{n-3}}$

$u_{n-2} = \sqrt{u_{n-3} u_{n-4}}$

$\vdots$

$u_5 = \sqrt{u_4 u_3}$

$u_4 = \sqrt{u_3 u_2}$

$u_3 = \sqrt{u_2 u_1}$

(Multiplying) $u_n \sqrt{u_{n-1}} = \sqrt{u_1 u_2^2}$

Let $\lim_{n\to\infty} u_n = l$ then $\lim_{n\to\infty} u_{n-1} = l$ also

$\therefore l \times \sqrt{l} = \sqrt{u_1 u_2^2}$

$\Rightarrow l^{\frac{3}{2}} = \sqrt{u_1 u_2^2} \Rightarrow l = \sqrt[3]{u_1 u_2^2}$

(24) $0 < u_1 < u_2$ $\therefore \frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$

$\Rightarrow \frac{1}{u_{n+2}} - \frac{1}{u_{n+1}} = \frac{1}{u_n} - \frac{1}{u_{n+2}}$

$\therefore \frac{1}{u_n} - \frac{1}{u_{n-1}} = \frac{1}{u_{n-2}} - \frac{1}{u_n}$

$\frac{1}{u_{n-1}} - \frac{1}{u_{n-2}} = \frac{1}{u_{n-3}} - \frac{1}{u_{n-1}}$

$\frac{1}{u_{n-2}} - \frac{1}{u_{n-3}} = \frac{1}{u_{n-4}} - \frac{1}{u_{n-2}}$

$\frac{1}{u_{n-3}} - \frac{1}{u_{n-4}} = \frac{1}{u_{n-5}} - \frac{1}{u_{n-3}}$

$\frac{1}{u_{n-4}} - \frac{1}{u_{n-5}} = \frac{1}{u_{n-6}} - \frac{1}{u_{n-4}}$

$\vdots$

$\frac{1}{u_5} - \frac{1}{u_4} = \frac{1}{u_3} - \frac{1}{u_5}$

$\frac{1}{u_4} - \frac{1}{u_3} = \frac{1}{u_2} - \frac{1}{u_4}$

$\frac{1}{u_3} - \frac{1}{u_2} = \frac{1}{u_1} - \frac{1}{u_3}$

$\therefore$ adding we get

$\frac{2}{u_n} + \frac{1}{u_{n-1}} = \frac{2}{u_2} + \frac{1}{u_1}$

Let $\lim_{n\to\infty} u_n = l$ then

also $\lim_{n\to\infty} u_{n-1} = l$

$\therefore \frac{2}{l} + \frac{1}{l} = \frac{2}{u_2} + \frac{1}{u_1}$

$\Rightarrow \frac{3}{l} = \frac{2}{u_2} + \frac{1}{u_1}$

$\Rightarrow l = \frac{3}{\frac{1}{u_1} + \frac{2}{u_2}}$