Charpit's Method:-

The fundamental idea in Charpit's method to solve the partial differential equ^n

$$f(x,y,z,p,q)=0 \quad \text{---}(1)$$

is to select a second ~~pordo~~ P.D.E of the first order $\phi(x,y,z,p,q,a)=0$ ---(2)

which contains an arbitrary constant a and which is such that

(i) equations (1) and (2) can be solved to give

$$p=p(x,y,z,a) \text{ and } q=q(x,y,z,a)$$

and (ii) the equ^n

$$dz = p(x,y,z,a)\,dx + q(x,y,z,a)\,dy \quad \text{---}(3)$$

is integrable with these values of p and q.

When such a function ϕ has been found, the sol^n of the equ^n (3), given by

$$f(x,y,z,a,b)=0 \quad \text{---}(4)$$

containing two arbitrary constants a, b will be a complete sol^n of ~~to~~ equ^n (1).

The corresponding subsidiary equ^n or Charpit's auxilliary equ^n, known as auxilliary equ^n ~~as~~ are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{d\phi}{0}.$$

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25	26	27	28	29	30	31

Now solving (1) and (2), the values of p and q are obtained and we can thus make equation (3) ~~integrable~~ integrable.

Then by usual method, we can find the complete integral of (1). Again, as usual, from complete integral, the general and singular integral can be found.

Solved Example :-

Q1. Solve by Charpit's method
$$px + qy = pq.$$

Sol:- The given equation is
$$f(x, y, z, p, q) = px + qy - pq = 0 \quad \text{--- (1)}$$

Charpit's auxilliary equ's are
$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or,
$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p + p.0} = \frac{dq}{q + q.0}$$

Taking the last two ratios we get
$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating, $p = aq$ where a is an arbitrary constant.

Putting the value of p in (1) we get

$$q = \frac{ax+y}{a} \text{ and}$$

hence $p = ax+y$.

Putting the values of p, q in $dz = p\,dx + q\,dy$, we get

$$dz = (ax+y)\,dx + \frac{ax+y}{a}\,dy.$$

or, $a\,dz = (ax+y)(a\,dx + dy)$

or, $a\,dz = (ax+y)\,d(ax+y)$.

Int, we get,

$$az = \frac{1}{2}(ax+y)^2 + c \quad \text{---} (2)$$

c being an arbitrary constant.

Which is the required complete integral.

writting $c = \phi(a)$, we get

$$az = \frac{(y+ax)^2}{2} + \phi(a) \quad \text{---} (3)$$

Differentiating w.r.t a we get

$$z = x(y+ax) + \phi'(a). \quad \text{---} (4)$$

Now elliminating a between (3) and (4), we get the general integral.

Again differentiating (2) w.r.t a and c we get

$z = x(y+ax)$ and $0 = 1$, which is impossible.

Hence there is no singular integral.

Q2. Find a complete integral of $z^2 = pqxy$.

Sol^n:- The given equation is

$$f(x,y,z,p,q) = z^2 - pqxy = 0. \quad \text{---} (1)$$

Charpit's auxilliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or, $\frac{dp}{-pqy+2pz} = \frac{dq}{-pqx+2qz} = \frac{dz}{-p(-qxy)-q(-pxy)} = \frac{dx}{qxy} = \frac{dy}{pxy}$. — (2)

Each part of (2)

$$= \frac{x\,dp+p\,dx}{x(-pqy+2pz)+pqxy} = \frac{y\,dq+q\,dy}{y(-pqx+2qz)+pqxy}$$

or, $\frac{x\,dp+p\,dx}{2pxz} = \frac{y\,dq+q\,dy}{2qyz}$

or, $\frac{d(xp)}{xp} = \frac{d(yq)}{yq}$

Integrating, we get

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$\log xp = \log yq + \log a^2$, a be an arbitrary constant.

or, $xp = a^2 yq$. — (3).

Solving (1) and (3) for p and q we get

$p = \frac{az}{x}$ and $q = \frac{z}{ay}$.

Putting the values of p, q in

$dz = p\,dx + q\,dy$ we get

$dz = \frac{az}{x}dx + \frac{z}{ay}dy$.

or, $\frac{1}{z}dz = \frac{a}{x}dx + \frac{1}{ay}dy$

Int., $\log z = a\log x + \frac{1}{a}\log y + \log c$, c being an arbitrary const.

or, $z = x^a y^{1/a} c$

which is the required complete integral.

Q3. Find the complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$.

Soln:- Here the equation is

$$f(x,y,z,p,q) = 2xz - px^2 - 2qxy + pq = 0 \quad (1)$$

Charpit's auxilliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or, $$\frac{dp}{2z - 2qy - 2px + p\cdot 2x} = \frac{dq}{-2qx + q\cdot 2x} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)}$$

$$= \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

or, $$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy + p}$$

From the second fraction, we get

$dq = 0$ i.e., $q = a$, a be a constant.

Putting $q = a$ in (1) we get

$$p = \frac{2xz - 2axy}{x^2 - a}$$

With these values of p and q in

$dz = p\,dx + q\,dy$ we get

$$dz = \frac{2xz - 2axy}{x^2 - a}\,dx + a\,dy$$

or, $$dz - a\,dy = \frac{2x(z - ay)}{x^2 - a}\,dx$$

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or, $\frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$

Int., $\log(z-ay) = \log(x^2-a) + \log c$, c be an arbitrary const.

or, $z - ay = c(x^2-a)$

or, $z = ay + c(x^2-a)$ — (2)

This is the complete integral of the given equ.

Diff^n (2) partially w.r.t. a and c, we get

$0 = y - c$ and $0 = x^2 - a$.

which gives, $a = x^2$ and $c = y$.

Putting the values of a and c in (2) we get

$z = x^2 y$

which satisfies the given equation and is the required singular integral of the equation.

Solve the following problem.

1. Find a complete integral of $(p+q)(px+qy) = 1$, by Charpit's method.

2. Find a complete integral of $xpq + yq^2 = 1$ by Charpit's method.

3. Find a complete integral of $z^2(p^2z^2+q^2) = 1$.