Second Order P.D.E.

A P.D.E with variable co-efficients will be called of second order, if it contains atleast one partial derivative of 2nd order and none of the order higher than the second.

With standard symbols

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x},$$
$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x},\ t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} \quad \text{--- (1)}$$

where z is a dependent variable and x and y are two independent variables,

A second order P.D.E can be written as

$$Rr + Ss + Tt + Pp + Qq + Zz = f(x, y) \quad \text{--- (2)}$$

where R, S, T, P, Q, Z are functions of x and y only and not all R, S, T are zero.

This equ^n may also be written as

$$Rr + Ss + Tt = V \quad \text{--- (3)}$$

where V is a fun^n of x, y, z, p and q.

The general sol^n of this equ^n will contain two arbitrary fun^{n}s.

When R, S, T are all constants, the equ^n (3) will be called a second order P.D.E with constant co-efficients.

An equⁿ $a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = 0$ —— (4)
where a, b, c are constants, will be called a second order ~~two~~ homogeneous partial differential equⁿ with ~~conds~~ constant co-efficients.

Classification of second order P.D.E :-

Let us consider a second order P.D.E, given by $Rr + Ss + Tt = f(x, y, z, p, q)$ —— (1)
where R, S, T are either functions of x and y or constants.
This equⁿ (1) will be called Elliptic, Hyperbolic or Parabolic

according as
$S^2 - 4RT < 0, > 0$ or $= 0$ respectively.
i.e, the equⁿ (1) will be
(i) Elliptic if $S^2 - 4RT < 0$
(ii) Hyperbolic if $S^2 - 4RT > 0$
and (iii) Parabolic if $S^2 - 4RT = 0$.

Solved Example :-

Q1. Determine the type of the equations
(a) $2\frac{\partial^2 z}{\partial x^2} - 8\frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} = 0$
(b) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$
(c) $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} + 3\frac{\partial z}{\partial y} = 0.$

Solⁿ:- (a) With usual notations
$R=2,\ S=-8,\ T=3$
Now, $S^2-4RT=(-8)^2-4.2.3=40>0$.
So this equⁿ is Hyperbolic.

(b) Here $R=1,\ S=-2,\ T=2$
Now, $S^2-4RT=(-2)^2-4.1.2=-4<0$
So this equⁿ is Elliptic.

(c) Here $R=1,\ S=-4,\ T=4$
Now $S^2-4RT=(-4)^2-4.1.4=0$
So this equⁿ is Parabolic.

Q2. Classify the following P.D.E.

(a) $2\left(\frac{\partial^2 z}{\partial x^2}\right)+4\left(\frac{\partial^2 z}{\partial x\partial y}\right)+3\frac{\partial^2 z}{\partial y^2}=2$.

(b) $\frac{\partial^2 z}{\partial x^2}+4\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2}=0$

(c) $xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$.

Solⁿ:- (a) With usual notations
$R=2,\ S=4,\ T=3$
Now $S^2-4RT=4^2-4.2.3=-8<0$.
So this equⁿ is Elliptic.

(b) Here, $R=1,\ S=4,\ T=4$
Now $S^2-4RT=4^2-4.1.4=0$
So this equⁿ is Parabolic.

(c) Here $R = xy$, $S = -(x^2 - y^2)$, $T = -xy$.

$$S^2 - 4RT = \{-(x^2-y^2)\}^2 - 4 \cdot xy \cdot (-xy)$$
$$= (x^2-y^2)^2 + 4x^2y^2$$
$$= (x^2+y^2)^2 > 0.$$

So this equ^n is Hyperbolic.

Problem:-

Q. Classify the types of the following equ^{n}s.

(a) $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial y^2} = 0$

(b) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial z}{\partial x} + z = 0$

(c) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} + 5\dfrac{\partial z}{\partial x} + z = 0.$

(d) $\dfrac{\partial^2 z}{\partial x^2} + x\dfrac{\partial^2 z}{\partial x \partial y} + y\dfrac{\partial^2 z}{\partial y^2} - xy\dfrac{\partial z}{\partial y} + z = 0.$

$\boxed{\text{Complete}}$