Partial Differential Equation.

Derivation of a Partial Differential Equⁿ.

Partial differential equⁿ can be formed in two ways: (a) by the elimination of arbitrary constants and (b) by the elimination of arbitrary functions.

1. Formation by eliminating arbitrary constants.

Let $\phi(x, y, z, a, b) = 0$. —① be any relation between two independent variables x, y and a dependent variable z and together with two arbitrary constants a and b. Diffⁿ ① partially w.r.t x and y, we get

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{or,} \quad \frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0. \quad ②$$

and
$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \text{or,} \quad \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0.$$

Now eliminating two quantities a and b from these relations ①, ② and ③, we get a P.D.E. of first order as

$$f(x, y, z, p, q) = 0.$$

Notes:- Standard notations are:

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y} \text{ and } t = \frac{\partial^2 z}{\partial y^2}.$$

2. By the elimination of two arbitrary functions.

Let us consider a relation between two functions u and v as given by $\phi(u,v)=0$. - - (1)
where u and are two functions of x, y, z; x, y being ~~to~~ two independent variables and z being a dependent variable.
Now, differentiating (1) partially w.r.t x and y, we get.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial \phi}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0.$$

or, $$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

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and $$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0.$$

Now eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (2) and (3) we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$$

or, $$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0$$

or, $$\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

which is of the form

$$Pp + Qq = R$$

where
$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

⊛ Solved Examples

Q1. Obtain a P.D.E by eliminating a and b from the equ^n $az + b = a^2x + y$.

Sol^n:- The equ^n is

$$az + b = a^2x + y \quad \text{--- (i)}$$

Diff^n (i) partially w.r.t x and y we get

$$a\frac{\partial z}{\partial x} = a^2 \quad \text{--- (ii)}$$

and
$$a\frac{\partial z}{\partial y} = 1 \quad \text{--- (iii)}$$

From (ii) and (iii), eliminating a, we get

$$\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = 1$$

i.e., $pq = 1$.

which is the required P.D.E.

Q2. Eliminate arbitrary constants from $z = (x-a)^2 + (y-b)^2$ to form the P.D.E.

Solⁿ:- Diffⁿ the given relation partially w.r.t. x and y seporately we get,

$$\frac{\partial z}{\partial x} \equiv p = 2(x-a) \quad \text{——— (i)}$$

and $$\frac{\partial z}{\partial y} \equiv q = 2(y-b) \quad \text{——— (ii)}$$

To eliminate a, b squaring and adding (1) and (2), we get,

$$p^2 + q^2 = 4[(x-a)^2 + (y-b)^2]$$
$$= 4z$$

or, $p^2 + q^2 = 4z$
which is the required P.D.E.

Q3. Obtain the PDE which has its general solⁿ $u = f(\sqrt{x^2+y^2})$, where f is an arbitrary function.

Solⁿ:- The equⁿ is $u = f(\sqrt{x^2+y^2})$ ——— (1)
Diffⁿ (1) partially w.r.t x we get

$$\frac{\partial u}{\partial x} = f'(\sqrt{x^2+y^2}) \cdot \frac{x}{\sqrt{x^2+y^2}} \quad \text{——— (ii)}$$

Diffⁿ (1) partially w.r.to y we get

$$\frac{\partial u}{\partial y} = f'(\sqrt{x^2+y^2}) \cdot \frac{y}{\sqrt{x^2+y^2}} \quad \text{——— (iii)}$$

From (ii) and (iii) we get

$$y\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

which is the required PDE.

Q4. Eliminate the arbitrary funⁿ ϕ and ψ from $z = \phi(x+iy) + \psi(x-iy)$, where $i^2=-1$.

Solⁿ:- Here the equⁿ is $z = \phi(x+iy)+\psi(x-iy)$ —(i)

Diffⁿ both sides of (i) partially w.r.t. x we get

$$\frac{\partial z}{\partial x} = \phi'(x+iy)+\psi'(x-iy)$$

again, diffⁿ w.r.t. x partially we get,

$$\frac{\partial^2 z}{\partial x^2} = \phi''(x+iy)+\psi''(x-iy) \quad \text{—(ii)}$$

Now diffⁿ both sides of (i) partially w.r.t y we get

$$\frac{\partial z}{\partial y} = i\,\phi'(x+iy) - i\,\psi'(x-iy)$$

Again diffⁿ w.r.t y partially we get

$$\frac{\partial^2 z}{\partial y^2} = i^2\phi''(x+iy) + i^2\psi''(x-iy)$$

$$= -\{\phi''(x+iy)+\psi''(x-iy)\} \quad [\because i^2=-1]$$

$$= -\frac{\partial^2 z}{\partial x^2}. \quad [\text{using (ii)}]$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

which is the required P.D.E.

① Solved the following problem.

1. Form the differential equation by eliminating a and b from
$z = (x^2+a)(y^2+b)$.

2. Eliminate a, b and c from
$z = a(x+y) + b(x-y) + abt + c$.

3. Form a partial differential equation by eliminating the fun
f from $z = xf\left(\frac{y}{x}\right)$.

4. Form a partial differential equation by eliminating the

arbitrary function from
$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$