1

Origin of Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown frunction of two or more independent variables is known as a *partial differential equation*.

For examples of partial differential equations we list the following:

$\partial z/\partial x+\partial z/\partial y=z+xy$ ... (1) $\qquad (\partial z/\partial x)^2+\partial^3 z/\partial y^3=2x(\partial z/\partial x)$ ... (2)

$z(\partial z/\partial x)+\partial z/\partial y=x$ ... (3) $\qquad \partial u/\partial x+\partial u/\partial y+\partial u/\partial z=xyz$ ... (4)

$\partial^2 z/\partial x^2=(1+\partial z/\partial y)^{1/2}$ ... (5) $\qquad y\{(\partial z/\partial x)^2+(\partial z/\partial y)^2\}=z(\partial z/\partial y)$ ... (6)

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$p=\partial z/\partial x$, $\quad q=\partial z/\partial y$, $\quad r=\partial^2 z/\partial x^2$, $\quad s=\partial^2 z/\partial x\partial y$ $\quad$ and $\quad t=\partial^2 z/\partial y^2$

In case there are n independent variables, we take them to be $x_1, x_2, \ldots, x_n$ and z is regarded as the dependent variable. In this case we use the following notations:

$p_1 = \partial z/\partial x_1$, $p_2 = \partial z/\partial x_2$, $p_3 = \partial z/\partial x_3$, and

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x\partial y$ and so on

1.7 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples. [Delhi Maths (H) 20...]

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y) z + S(x, y)$

For examples, $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + ...$

are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = ...$

are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $x^2zp + y^2zp = xy$ and $(x^2 - yz)\, p + (y^2 - zx)\, q = ...$

are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $p^2 + q^2 = 1$, $p\,q = z$ and $x^2p^2 + y^2q^2 ...$

are all non-linear partial differential equations.

1.8 Origin of partial differential equations.

We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

1.9 Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation

$$F(x, y, z, a, b) = 0,$$

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$\partial F/\partial x + p(\partial F/\partial z) = 0$ and $\partial F/\partial y + q(\partial F/\partial z) = 0$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0,$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$f(x, y, z, a, b) = 0$, $\quad \partial F/\partial x = 0 \quad$ and $\quad \partial F/\partial y = 0$

...equation free from a and b will be the required partial differential equation.

Three situations may arise :

Situation I. When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.

For example, consider $\quad z = ax + y, \quad$ ... (1)

a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $\quad \partial z/\partial x = a \quad$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\quad \partial z/\partial y = 1 \quad$ ... (3)

Eliminating a between (1) and (2) yields $\quad z = x(\partial z/\partial x) + y \quad$ ... (4)

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.

Example: *Eliminate a and b from* $\quad az + b = a^2x + y \quad$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$a(\partial z/\partial x) = a^2 \quad$ ... (2) $\qquad a(\partial z/\partial y) = 1 \quad$ ... (3)

Eliminating a from (2) and (3), we have $\quad (\partial z/\partial x)(\partial z/\partial y) = 1$,

which is the unique partial differential equation of order one.

Situation III. When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.

Example: *Eliminate a, b and c from* $\quad z = ax + by + cxy \quad$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$\partial z/\partial x = a + cy \quad$ ... (2) $\qquad \partial z/\partial y = b + cx \quad$ ... (3)

From (2) and (3), $\quad \partial^2 z/\partial x^2 = 0, \qquad \partial^2 z/\partial y^2 = 0 \quad$ ... (4)

$$\partial^2 z/\partial x\partial y = c \quad \text{... (5)}$$

Now (2) and (3) $\Rightarrow$ $\quad x(\partial z/\partial x) = ax + cxy \quad$ and $\quad y(\partial z/\partial y) = by + cxy$

$$x(\partial z/\partial x) + y(\partial z/\partial y) = ax + by + cxy + cxy$$

$$x(\partial z/\partial x) + y(\partial z/\partial y) = z + xy(\partial^2 z/\partial x\partial y), \text{ using (1) and (5)} \quad \text{... (6)}$$

Thus, we get three partial differential equations given by (4) and (6), which are all of order ...

SOLVED EXAMPLES BASED ON RULE I OF ART 1.9

Ex. 1. *Find a partial differential equation by eliminating a and b from $z = ax + by + a^2 + b^2$.*

Sol. Given $\quad z = ax + by + a^2 + b^2. \quad$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a \quad$ and $\quad \partial z/\partial y = b$.