Math Sept. class notes by Amalansu sekhar Pattanayak. Page-1

For 2nd Sem (H), Differential Equation Unit-3 Core-4.

Phase Plane / Equilibrium points

[For Notes on 2nd ~~Phase~~ term on (29.04.2020) for finding P.I. of $\frac{1}{f(D)} \cdot e^{ax}$ if $f(D) = (D-a)^r \cdot \phi(D)$ on Page-4 in the bottom.

then $P.I = \frac{1}{(D-a)^r \phi(D)} \cdot e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \cdot \frac{x^r}{r!}$

(Rectify the earlier by that) where ~~for~~ $\phi(a) \neq 0$]

Rectify it [on the Page-4 in the bottom. {if is $\phi(a)=0$ wrong}

Phase ~~Plane~~ Plane

~~The~~ The state of dynamical system changes with time. Continuous dynamical sytems are usually represented by a differential equation like $\frac{dx}{dt} = X(x)$ (1) where x denotes the state ~~of~~ or position of the system. (1) is like the form $\frac{dx}{dt} = \mu x$ is an autonomous D.E. Since the independent variable t does not appear on the right hand side. For an initial value (1) represents the set of solutions. For increasing time t there will be a path or orbit or trajectory. The set of all trajectories of a flow is called a Phase Portrait. A phase Portrait in two dimensions is called a Phase Plane.

Let us suppose that the non linear equation of the form $\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt})$ describes a dynamical system. The state of this system at time t is determined by the values of x and $\frac{dx}{dt}$. The Plane of the variables x and $\frac{dx}{dt}$ is called a Phase plane.

If we put $y = \frac{dx}{dt}$, then the above equation is equivalent to the system of equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = F(x, y)$$

Now the autonomous system $\frac{dx}{dt} = f(x)$ where $f(x)$ is independent of t. Then it becomes of the form (for linear)

$$\frac{dx}{dt} = Ax$$ where A is an $n \times n$ matrix with constant elements. Let us study the phase plane the dynamical system of the form

$$\frac{dx}{dt} = F(x, y) \quad \cdots (1)$$
$$\frac{dy}{dt} = G(x, y) \quad \cdots (2)$$

The paths of (1) and (2) cover the entire phase plane and do not intersect one another. The points where both $F(x,y)$ and $G(x,y)$ vanish are called critical points or fixed points or singular points or equilibrium points of the system. If (x_0, y_0) be such a point that $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$ then the unique solution is $x = x_0$, and $y = y_0$.

If there are no such other critical points except (x_0, y_0) within the circle of centre (x_0, y_0), then this critical point (x_0, y_0) of this system is said to be isolated.

Let us consider the linear plane autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1 x + b_1 y \\ \frac{dy}{dt} &= a_2 x + b_2 y \end{aligned}\right\}$$ where a_1, b_1, a_2, b_2 are real constants

This can be written in the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$ where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$ and $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0$.

So that $(0,0)$ is only critical point.

Then the characteristic equⁿ of A is

$$\begin{vmatrix} a_1-\lambda & b_1 \\ a_2 & b_2-\lambda \end{vmatrix} = 0 \quad \text{or,} \quad \lambda^2-(a_1+b_2)\lambda+(a_1b_2-a_2b_1)=0 \quad (3)$$

Let λ_1, λ_2 be the roots of equⁿ (3). These are the eigen values of A. The nature of critical point is determined by the nature of the eigen values λ_1, λ_2 :

1) If λ_1, λ_2 be real, distinct and of the same sign (or λ_1 and λ_2 be real and equal) then Nature of critical point of this system is node. This node
   a) if λ_1, λ_2 both be negative critical point (0,0) is Stable
   b) if λ_1, λ_2 both be +ve " " is unstable

2) If λ_1, λ_2 be real, unequal and of ~~same~~ opposite sign then Nature of critical Point of linear system is Saddle Point. ~~It is unstable~~ It is unstable.

3) λ_1, λ_2 roots are Conjugate complex but not pure imaginary then Nature of the System is Spiral Point or focal point ⇒ ① It is stable if the real part is negative. ② Unstable if real part is Positive.

④ If the roots are pure imaginary Nature of the critical point (0,0) of this system is a Centre. ~~It is stable, not asymptotically~~

Ex ① Find the Phase curve of the dynamical system equations | or the general Solution or paths

$$\dot{x} = \frac{dx}{dt} = y$$

$$\dot{y} = \frac{dy}{dt} = -5x + 2y.$$

Also describe the nature of the stationary point (critical point)

Sol^n:- The matrix form of the linear dynamical system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Here $(0,0)$ is only the equillibrium point as $\frac{dx}{dt} = y = 0$ so $\dot{y} = 0$ gives $x = 0$

So the characteristic equation is

$$\begin{vmatrix} -\lambda & 1 \\ -5 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 5 = 0.$$

$$\text{or } \lambda = \frac{2 \pm \sqrt{4 - 4.5}}{2}$$

$$\therefore \lambda = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

roots are conjugate with real $\lambda > 0$

Hence the equilibrium or stationary point is an unstable spiral point.

Equation of phase curve or general sol^n is

$$x = e^t(A\cos 2t + B\sin 2t)$$

$$\dot{x} = y = e^t(A\cos 2t + B\sin 2t) + e^t(-2A\sin 2t + 2B\cos 2t)$$

$$= e^t\{(A+2B)\cos 2t + (B - 2A)\sin 2t\}$$

Ex ② Discuss the nature of the equilibrium points of a damped oscillator as given by the equation $\frac{d^2x}{dt^2} + 2\mu\frac{dx}{dt} + \lambda^2 x = 0$; $\mu > 0$.

Sol^n: The given equation is equivalent to the autonomous system

$$① \begin{cases} \frac{dx}{dt} = \dot{x} = y \quad \text{and} \quad \left(\text{as } \frac{dx}{dt} = y\right) \\ \frac{dy}{dt} = \frac{d^2x}{dt^2} = -\lambda^2 x - 2\mu y \end{cases}$$

Here $(0,0)$ is the only equilibrium or critical point.

characteristic equation of ① is

$$\begin{vmatrix} 0-\lambda & 1 \\ -n^2 & -2\mu-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2+2\mu\lambda+n^2=0$$

giving $\lambda = -\mu + \sqrt{\mu^2-n^2}$

Thus the equilibrium point is

1) a stable node for $\mu > n > 0$
2) an unstable node for $\mu < -n < 0$
3) a stable spiral point for $0 < \mu < n$
4) an unstable spiral point for $-n < \mu < 0$

If $\mu = n$, then it a node

Here $\dfrac{dy}{dx} = -\dfrac{n^2x+2\mu y}{y}$ and $(0,0)$ is the equilibrium point.

In case of light damping when $\mu < n$, this equilibrium point is called focal point and in case of heavy damping when $\mu > n$, then it is nodal point.

EX ① Determine the nature of the critical point of the system $\dfrac{dx}{dt} = 2x - 7y$; $\dfrac{dy}{dt} = 3x - 8y$

Do it yourself : Hints dynamical system is $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ find characteristic Equ^n as

$\begin{vmatrix} 2-\lambda & 7 \\ 3 & -8-\lambda \end{vmatrix} = 0$ find the value of λ and find the nature and the ~~path of~~ path or Trajectory / orbit

EX ② Show that the equilibrium point of the linear dynamical system $\dfrac{dx}{dt} = x+3y$, $\dfrac{dy}{dt} = 3x+y$ is a saddle Point that of the system $\dfrac{dx}{dt} = 3x-2y$, $\dfrac{dy}{dt} = 5x-3y$ is a Centre.