E - LEARNING MATERIALS

SEM - 4, CC - 8, Unit - 5

Topic - Power Series

Prepaired by - Dr. Alauddin Dafadar.

Definition:- R is called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. The open interval $(-R, R)$ is called the interval of convergence of the series.

Note: In case of power series which is every where convergent the radius of convergence is infinity ie $R = \infty$, and for power series which is no where convergent the radius of convergence is zero ie $R = 0$.

Determination of radius of convergence.

CH.99:- Cauchy-Hadamard theorem:

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and $\overline{\lim} \sqrt[n]{|a_n|} = \mu$, then

2000 (i) if $\mu = 0$, the series is every where convergent

(ii) if $0 < \mu < \infty$, then the series is absolutely convergent for all real x satisfying $|x| < \frac{1}{\mu}$ and is divergent for all real x satisfying $|x| > \frac{1}{\mu}$.

(iii) if $\mu = \infty$ then the series is no where convergent.

Proof:-

(ii)

Let $u_n = a_n x^n$

$\therefore |u_n| = |a_n||x^n|$

or $|u_n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}|x|$

$\overline{\lim} |u_n|^{\frac{1}{n}} = \overline{\lim} |a_n|^{\frac{1}{n}}|x|$

$= \mu|x|$

If $|x| < \frac{1}{\mu}$ then $\overline{\lim} |u_n|^{\frac{1}{n}} < 1$

By Cauchy's root test $\sum |u_n|$ is convergent if $|x| < \frac{1}{\mu}$

ie $\sum u_n$ is absolutely convergent if $|x| < \frac{1}{\mu}$.

ie the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $|x| < \frac{1}{\mu}$.

If $|x| > \frac{1}{\mu}$ then $\lim |u_n|^{\frac{1}{n}} > 1$

$\therefore \lim |u_n| \neq 0$.

Therefore the sequence can't converge to 0 i.e. $\lim u_n \neq 0$

i.e. the series $\sum u_n$ is not convergent

i.e. the power series is not convergent for all x satisfying $|x| > \frac{1}{\mu}$.

Note :- the radius of convergence of the power series is $\frac{1}{\mu}$.

Ratiotest : Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and let

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \mu$ then

(i) if $\mu = 0$, the series is everywhere convergent

(ii) if $0 < \mu < \infty$, then the series is absolutely convergent for all real x satisfying $|x| < \frac{1}{\mu}$ and is divergent for all real x satisfying $|x| > \frac{1}{\mu}$.

(iii) if $\mu = \infty$, the series is nowhere convergent.

Note : The radius of convergence is $\frac{1}{\mu}$.

C.H. Find the radius of convergence of the power series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$$

$\Rightarrow$ Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_n = \frac{n^n}{n!}$ for all $n \geq 1$,

$a_0 = 0$.

$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^{n+1}}{(n+1)} \cdot \frac{1}{n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n$

for all $n \geq 1$.

$\therefore \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left(1+\frac{1}{n}\right)^n = e$.

$\therefore$ The radius of convergence is $\frac{1}{e}$.

C.H, C.H University? Find the radius of convergence of the power series

$$\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots$$

$\Rightarrow$ Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = \frac{1}{3}$, $a_1 = -1$,

$a_2 = \frac{1}{3^2}$, $a_3 = -1$,

$a_4 = \frac{1}{3^4}$, $a_5 = -1$,

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$\lim |a_{2n-1}|^{\frac{1}{2n-1}} = 1$ and $\lim |a_{2n}|^{\frac{1}{2n}} = \lim \left|\frac{1}{3^{2n}}\right|^{\frac{1}{2n}} = \frac{1}{3}$

$\therefore \overline{\lim} |a_n|^{\frac{1}{n}} = 1$.

$\therefore$ the radius of convergence is 1.

C.H. C.H. C.H University Find the radius of convergence of the power series

$$x + \frac{(\lfloor 2)^2}{\lfloor 4} x^2 + \frac{(\lfloor 3)^2}{\lfloor 6} x^3 + \cdots$$

$\Rightarrow$ Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 0$, $a_1 = 1$, $a_2 = \frac{(\lfloor 2)^2}{\lfloor 4}$

$\therefore \; a_n = \frac{(\lfloor n)^2}{\lfloor 2n} \quad \forall n \geq 2$

Now $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(\lfloor n+1)^2}{\lfloor 2(n+1)} \cdot \frac{\lfloor 2n}{(\lfloor n)^2}$ for all $n \geq 2$

$= \frac{n+1}{2(2n+1)}$ for all $n \geq 2$

$\therefore \lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left[\frac{n+1}{2(2n+1)}\right] = \frac{1}{4}$.

$\therefore$ the radius of convergence is $\frac{1}{\frac{1}{4}} = 4$.

C.H.-C.H.-C.H.C.H

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{1}{3^n}$, if n be odd

$= \frac{1}{2^n}$, if n be even.

$\Rightarrow$ When n is odd $\lim|a_n|^{\frac{1}{n}} = \frac{1}{3}$

When n is even $\lim|a_n|^{\frac{1}{n}} = \frac{1}{2}$

$\therefore \; \overline{\lim}|a_n|^{\frac{1}{n}} = \frac{1}{2}$

$\therefore$ the radius of convergence is $\frac{1}{\frac{1}{2}} = 2$.

CH.CH.CH.CH.CH

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_0 = 1$, $2 \leq |a_n| \leq 3$ for all $n \geq 1$.

Proof: $2 \leq |a_n| \leq 3$ for $n \geq 1$.

or $2^{\frac{1}{n}} \leq |a_n|^{\frac{1}{n}} \leq 3^{\frac{1}{n}}$ for $n \geq 1$

Now $\lim 2^{\frac{1}{n}} = 1$ and $\lim 3^{\frac{1}{n}} = 1$.

By sandwich theorem $\lim|a_n|^{\frac{1}{n}} = 1$