E - LEARNING MATERIALS

SEM - 4, CC - 8, Unit - 5

Topic - Power Series

Prepared by - Dr. Alauddin Dafadar

$\therefore$ the radius of convergence is $\frac{1}{1} = 1$. (1)

CH.CH.CH.CH.CH.CH

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = 2^n + 3^n$.

$\Rightarrow$

Let us find the radius of convergence of the power series $\sum_{n=0}^{\infty} b_n x^n$ where $b_n = 2^n$.

Here $\lim \left| \frac{b_{n+1}}{b_n} \right| = \lim \frac{2^{n+1}}{2^n} = 2$

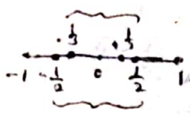
$\therefore$ the radius of convergence of the power series is $\frac{1}{2}$.

Let us find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ where $c_n = 3^n$.

Here $\lim \left| \frac{c_{n+1}}{c_n} \right| = \lim \frac{3^{n+1}}{3^n} = 3$.

$\therefore$ the radius of convergence of the power series is $\frac{1}{3}$.

The given series $\sum_{n=0}^{\infty} a_n x^n$ is the sum of two power series $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$.



The common region of the convergence of two power series $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$ is $-\frac{1}{3} < x < \frac{1}{3}$.

$\therefore$ the radius of convergence of the given series is $\frac{1}{3}$.

CH. 2001 ** ** ** ** ** ** ** ** V.V.I

3

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Then the series is uniformly convergent on $[-s, s]$ where $0 < s < R$.

Proof :- Let $f_n(x) = a_n x^n$, $n \geq 0$

Since R is the radius of convergence of the power

Series, the series is absolutely convergent for all x satisfying $|x| < R$.

Let us choose a +ve real number s such that $0 < s < R$

Then the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x satisfying $|x| \le s$

~~By weierstrass M-test~~

Therefore $\sum_{n=1}^{\infty} |a_n s^n|$ is convergent

Let $M_n = |a_n s^n|$ for all $n \in N$

Then $\sum M_n$ is a convergent series of +ve real numbers and

for all $n \in N$ $|f_n(x)| \le M_n$ for all $x \in [-s, s]$, since $|f_n(x)| = |a_n x^n| \le |a_n s^n|$ for all x satisfying $|x| \le s$.

By weierstrass M-test $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[-s, s]$.

Hence the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-s, s]$ — (I)

Version 1 TH 2001

Let $R > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Then the series is uniformly convergent on $[-R+\epsilon, R-\epsilon]$ where ϵ is a small +ve number.

Proof: Since $\epsilon > 0$ is arbitrarily small, $R - \epsilon > 0$.

Let $s = R - \epsilon$

Then $0 < s < R$ and therefore the power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-s, s]$ i.e. on $[-R+\epsilon, R-\epsilon]$.

NO 2000 Version 2 NO 2000

Let $R > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. If $[a, b]$ be any closed interval contained in $(-R, R)$ then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[a, b]$.

Proof :-

Let us choose $\epsilon > 0$ such that

$R - \epsilon > 0$ $\forall$ $-R < -R+\epsilon < a < b < R-\epsilon < R$

Let $R - \epsilon = s$ then $0 < s < R$ and

$-R < -s < a < b < s < R$

Since the power series is uniformly convergent on $[-s, s]$ and $[a, b] \subset [-s, s]$ the power series is uniformly convergent on $[a, b]$.

for examination V.V.I Thm. *******

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Let $f(x)$ be the sum of the series on $(-R, R)$. Then f is continuous on $(-R, R)$.

Proof : Since R is the radius of convergence of the power series, the series is uniformly convergent on $[-R+\delta, R-\delta]$ where δ is an arbitrarily small +ve number

[illegible] actually [illegible]

S_0 is not defined $S_1 = f_0 + f_1$

Let $f_n(x) = a_n x^n$, $n \geq 0$

Let $S_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x)$, for $n \geq 1$ Imp

corelated

v.v.I

Since the series is uniformly convergent on $[-R+\delta, R-\delta]$ to the function f, the sequence $\{S_n\}$ is uniformly convergent on $[-R+\delta, R-\delta]$.

Let $c \in [-R+\delta, R-\delta]$ [illegible]

Let us choose $\epsilon > 0$.

[illegible] limit [illegible] $N'(c,\delta)$ Continuity $N(c,\delta')$

(4)

There exist a natural number K such that for all $x \in [-R+\delta, R-\delta]$, $|S_n(x) - f(x)| < \frac{\epsilon}{3}$ $\forall n \geq K$

Hence for all $x \in [-R+\delta, R-\delta]$, $|S_K(x) - f(x)| < \frac{\epsilon}{3}$ for all x

$\therefore |S_K(c) - f(c)| < \frac{\epsilon}{3}$.

Since each f_n is continuous at c, S_n is continuous at c for $n \geq 1$.

Therefore there exist a +ve δ' such that ~~not δ~~

$|S_K(x) - S_K(c)| < \frac{\epsilon}{3}$ $\forall x \in N(c,\delta') \cap [-R+\delta, R-\delta]$

[illegible] $N'(c$ [illegible] $N(c$ [illegible]

we have not f

$$|f(x) - f(c)| = |f(x) - S_K(x) + S_K(x) - S_K(c) + S_K(c) - f(c)|$$

$$\leq |f(x) - S_K(x)| + |S_K(x) - S_K(c)| + |S_K(c) - f(c)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \qquad \forall x \in N(c,\delta') \cap [-R+\delta, R-\delta]$$

or $|f(x) - f(c)| < \epsilon$ $\forall x \in N(c,\delta') \cap [-R+\delta, R-\delta]$

This proves that f is continuous at c.

Since c is arbitrary, f is continuous on $[-R+\delta, R-\delta]$.

Since δ is arbitrary, f is continuous on $(-R, R)$.

A power series can be integrated term-by term on any closed and bounded interval contained with in the interval of convergence.

$\Rightarrow$

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R and let $f(x)$ be the sum of the series.

The theorem states that for any closed interval $[a, b] \subset (-R, R)$,

$$\int_a^b a_0\,dx + \int_a^b a_1 x\,dx + \int_a^b a_2 x^2\,dx + \cdots = \int_a^b f(x)\,dx$$