Real Sequence 1

Sequence: Let S be a set. A mapping $f:\mathbb{N}\to S$ is said to be a sequence in S. The symbol $\{f(n)\}$ is used to denote a sequence. Also the symbols $\{f(1), f(2), f(3), ...\}$ is used to denote the sequence f. $f(n)$ is called the nth term of the sequence. Then numbers of terms in a sequence is always infinite.

If $S=\mathbb{C}$ then the mapping $f:\mathbb{N}\to S$ is said to be a **complex sequence**. For example, $f(n)=i^n, n\in\mathbb{N}$. That is, $f(1)=i$, $f(2)=-1, f(3)=-i$, $f(4)=1$, $f(5)=i$ etc. It is denoted by $\{i^n\}$ or $\{i,-1,-i,1,i,.......\}$

If $S=\mathbb{R}$ then the mapping $f:\mathbb{N}\to S$ is said to be a **sequence in** $\mathbb{R}$, or a **real sequence**. For example $f(n)=n, n\in\mathbb{N}$ that is $f(1)=1, f(2)=2, f(3)=3,.........$ the sequence is denoted by $\{n\}$ or $\{1,2,3,4,.......\}$

The symbols like $\{u_n\}$, $\{s_n\}$, $\{x_n\}$, etc. shall also be used to denote a sequence.

The set $\{f(n): n\in\mathbb{N}\}$ is called the **range of the real sequence** $\{f(n)\}$.

Examples:

1: $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},......\right\}$ is a sequence which is called harmonic sequence. The sequence can be written as $\{f(n)\}$ where $f(n)=\frac{1}{n}$.

2: The sequence $\{2n-1\}$ is the sequence of all natural odd numbers which is $\{1,3,5,7,......\}$

3: Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=n^2, n\in\mathbb{N}$. The sequence is $\{n^2\}$. It is also denoted by $\{1^2,2^2,3^2,..............\}$.

4: The range of the sequence $\{(-1)^n\}$ is $\{-1,1\}$ because the elements of the sequence are 1 and -1. The sequence is also denoted by $\{-1,1,-1,1,............\}$.

6: Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=(-1)^n n, n\in\mathbb{N}$. The sequence is denoted by $\{(-1)^n n\}$. It is also denoted by $\{-1,2,-3,4,..............\}$.

7: Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=(-1)^n\frac{1}{n}, n\in\mathbb{N}$. The sequence is denoted by $\left\{(-1)^n\frac{1}{n}\right\}$. It is also denoted by $\left\{-1,\frac{1}{2},-\frac{1}{3},\frac{1}{4},............\right\}$.

8. The sequence $\{x_n\}$ where $x_n=\frac{1}{2}\{1+(-1)^n\}$ is given by $\{0,1,0,1,0,......\}$. Therefore the range of the sequence is given by $\{0,1\}$

Constant sequence: Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=c$, $n\in\mathbb{N}$ where c is a fixed real number. Then the sequence $\{f(n)\}$ is said to be a constant sequence.

Bounded sequence: A real sequence $\{u_n\}$ is said to be **bounded above** if there exists a real number G such that $u_n \le G$ for all $n \in \mathbb{N}$. G is said to be an **upper bound** of the sequence $\{u_n\}$.

A real sequence $\{u_n\}$ is said to be **bounded below** if there exists a real number g such that $f(n) \ge g$ for all $n \in \mathbb{N}$. g is said to be a **lower bound** of the sequence $\{u_n\}$.

A real sequence $\{u_n\}$ is said to be **bounded sequence** if there exist real numbers g, G such that $g \le u_n \le G$ for all $n \in \mathbb{N}$.

Supremum and infimum of a sequence:

For a real sequence $\{u_n\}$ bounded above, the range of the sequence is a set bounded above and by the supremum property of $\mathbb{R}$, the range set has the least upper bound, which is also called the least upper bound of the sequence $\{u_n\}$ and is denoted by $\sup\{u_n\}$. Also for a real sequence $\{u_n\}$ bounded below, there exists a greatest lower bound and is denoted by $\inf\{u_n\}$.

Properties of supremum and infimum of a sequence: The least upper bound of a real sequence $\{u_n\}$ is a real number M satisfying the following conditions:

(i) $f(n) \le M$ for all $n \in \mathbb{N}$.

(ii) for each pre-assigned positive ε, there exists a natural number k such that $f(k) > M - \varepsilon$.

The greatest lower bound of a real sequence $\{u_n\}$ is a real number m satisfying the following conditions:

(i) $f(n) \ge m$ for all $n \in \mathbb{N}$,

(ii) for a pre-assigned positive ε, there exists a natural number k such that $f(k) < m + \varepsilon$.

For a real sequence $\{u_n\}$ unbounded above, we define $\sup\{u_n\} = \infty$.

For a real sequence $\{u_n\}$ unbounded below, we define $\inf\{u_n\} = -\infty$.

Examples

1: The sequence $\left\{\frac{1}{n}\right\}$ is a bounded sequence. 0 is the greatest lower bound and 1 is the least upper bound of the sequence.

2: The sequence $\{n^2\}$ is bounded below and unbounded above. Here $\sup\{u_n\} = \infty$, $\inf\{u_n\} = 1$.

Ex 1: Find $\sup\{u_n\}$ and $\inf\{u_n\}$ where (i) $u_n = (-1)^n + \cos\frac{n\pi}{4}$, (ii) $u_n = \frac{(-1)^n}{n} + \sin\frac{n\pi}{2}$.

✍ The sequence $\{u_n\}$ is $\left\{-1+\frac{1}{\sqrt{2}}, 1, -1-\frac{1}{\sqrt{2}}, 0, -1-\frac{1}{\sqrt{2}}, 1, -1+\frac{1}{\sqrt{2}}, 2, 1+\frac{1}{\sqrt{2}}, \dots\dots\dots\right\}$.

Therefore $\sup\{u_n\} = 2$ and $\inf\{u_n\} = -1-\frac{1}{\sqrt{2}}$.

(ii) The sequence $\{u_n\}$ is $\left\{0, \frac{1}{2}, -\frac{1}{3}-1, \frac{1}{4}, -\frac{1}{5}+1, \frac{1}{6}, -\frac{1}{7}-1, \frac{1}{8}, -\frac{1}{9}+1, \dots\dots\dots\right\}$.

Therefore $\sup\{u_n\} = 1$ and $\inf\{u_n\} = -\frac{1}{3}-1$.

Limit of a sequence: Let $\{u_n\}$ be a sequence. A real number l is said to be a limit of the sequence $\{u_n\}$ if corresponding to a pre-assigned positive ε there exists a natural number k (depending on ε) such that $|f(n)-l|<\varepsilon$ for all $n \geq k$.

i.e., $l-\varepsilon < f(n) < l+\varepsilon$ for all $n \geq k$

Theorem: A sequence can have at most one limit.

Proof: If possible, let a sequence $\{u_n\}$ have two distinct limits l_1 and l_2 where $l_1 < l_2$.

Let $\varepsilon = \frac{1}{2}(l_2 - l_1)$. Then $\varepsilon > 0$ and $l_1 + \varepsilon = l_2 - \varepsilon$. Therefore the neighbourhoods $(l_1 - \varepsilon, l_1 + \varepsilon)$ and $(l_2 - \varepsilon, l_2 + \varepsilon)$ are disjoint.

Since l_1 is a limit of the sequence, for this chosen ε, there exists a natural number k_1 such that $l_1 - \varepsilon < u_n < l_1 + \varepsilon$ for all $n \geq k_1$.

Since l_2 is a limit of the sequence, for this chosen ε, there exists a natural number k_2 such that $l_2 - \varepsilon < u_n < l_2 + \varepsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $l_1 - \varepsilon < u_n < l_1 + \varepsilon$ and $l_2 - \varepsilon < u_n < l_2 + \varepsilon$ for all $n \geq k$

This cannot happen since the neighbourhoods $(l_1 - \varepsilon, l_1 + \varepsilon)$ and $(l_2 - \varepsilon, l_2 + \varepsilon)$ are disjoint.

Therefore our assumption that $l_1 \neq l_2$ is wrong.

Hence $l_1 = l_2$

Convergent sequence: A real sequence $\{u_n\}$ is said to be convergent to a real number l if for a pre-assigned positive ε there exists a natural number k such that $|f(n)-l|<\varepsilon$ for all $n \geq k$.

i.e., $l-\varepsilon < u_n < l+\varepsilon$ for all $n \geq k$

In this case we write $\lim_{n\to\infty} u_n = l$.

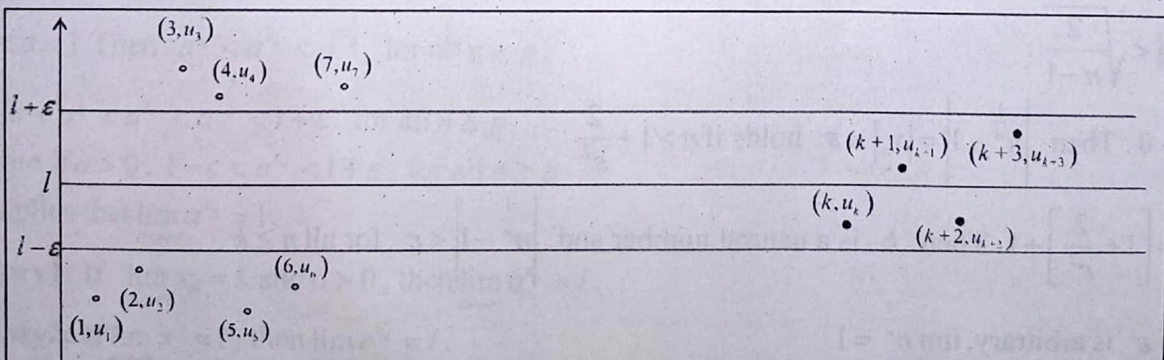
Divergent sequence: A sequence $\{u_n\}$ is said to de divergent if it is not convergent.

Geometrical Interpretation:

If the sequence $\{u_n\}$ converge to l then for every pre assigned +ve ε there exists a natural number k s.t. $|u_n - l| < \varepsilon$ for all $n \geq k$

i.e. $l-\varepsilon < u_n < l+\varepsilon$ for all $n \geq k$

Geometrically this fact can be described as after some certain stage all members of $\{u_n\}$ lies



inside the boundary of length 2ε whose centre is the line $y=l$.

Ex 2: Show that the sequence $\left\{\frac{n^2+1}{n^2}\right\}$ converges to 1.

✍ Let us choose a positive ε.

Now $\left|\frac{n^2+1}{n^2}-1\right|<\varepsilon$ will hold if $\frac{1}{n^2}<\varepsilon$, i.e., if $n>\frac{1}{\sqrt{\varepsilon}}$.

Let $k=\left[\frac{1}{\sqrt{\varepsilon}}\right]+1$. Then k is a natural number and $\left|\frac{n^2+1}{n^2}-1\right|<\varepsilon$ for all $n\geq k$.

This proves that $\lim\limits_{n\to\infty}\frac{n^2+1}{n^2}=1$.

Ex 3: Let $u_n=2$ for all $n\in\mathbb{N}$. Show that the sequence $\{u_n\}$ converges to 2.

✍ Let us choose a positive ε.

Now $|u_n-2|=0<\varepsilon$ holds for all $n\geq 1$.

Therefore $\lim\limits_{n\to\infty}u_n=2$

Ex 4: Show that $\lim\limits_{n\to\infty}n^{\frac{1}{n}}=1$.

Or,

Show that the sequence $\left\{n^{\frac{1}{n}}\right\}$ converge to 1 **VU'2004, 08**

✍ $n^{\frac{1}{n}}>1$ for all $n>1$.

Let $n^{\frac{1}{n}}=1+x_n$ where $x_n>0$.

Then $n=(1+x_n)^n$

$$=1+nx_n+\frac{n(n-1)}{2}x_n^{\ 2}+.............+x_n^{\ n}$$

$$>\frac{1}{2}n(n-1)x_n^{\ 2}.$$

Clearly, $x_n^{\ 2}<\frac{2}{n-1}$ for all $n>1$

Or, $|x_n|<\sqrt{\frac{2}{n-1}}$.

Let $\varepsilon>0$. Then $\left|n^{\frac{1}{n}}-1\right|=|x_n|<\varepsilon$ holds if $n>1+\frac{2}{\varepsilon^2}$

Let $k=\left[1+\frac{2}{\varepsilon^2}\right]+1$. Then k is a natural number and $\left|n^{\frac{1}{n}}-1\right|<\varepsilon$ for all $n\geq k$.

Since ε is arbitrary, $\lim\limits_{n\to\infty}n^{\frac{1}{n}}=1$

Ex 5: Show that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ if $a > 0$.

✍ **Case1:** $a = 1$. In this case the sequence converges to 1.

Case2: $a > 1$. Then $a^{\frac{1}{n}} > 1$. Let $a^{\frac{1}{n}} = 1 + x_n$ where $x_n > 0$.

Then $a = (1 + x_n)^n > 1 + nx_n$ for $n > 1$

Let $\varepsilon > 0$. Then $\left| a^{\frac{1}{n}} - 1 \right| < \varepsilon$ holds if $\frac{a-1}{n} < \varepsilon$ i.e., if $n > \frac{a-1}{\varepsilon}$.

Let $k = \left[\frac{a-1}{\varepsilon}\right] + 1$. Then k is a natural number and $\left| a^{\frac{1}{n}} - 1 \right| < \varepsilon$ for all $n \geq k$.

Since ε is arbitrary, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$.

Case3: $0 < a < 1$. Let $b = \frac{1}{a}$. Then $b > 1$ and $\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{b^{\frac{1}{n}}} = 1$, by case2

Combining the cases, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ if $a > 0$.

Ex 6: If $\lim_{n\to\infty} x_n = 0$ and $a > 0$, then show that $\lim_{n\to\infty} a^{x_n} = 1$.

✍ We have $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ and $\lim_{n\to\infty} a^{-\frac{1}{n}} = 1$.

Let us choose $\varepsilon > 0$. There exists natural numbers k_1, k_2 such that $1 - \varepsilon < a^{\frac{1}{n}} < 1 + \varepsilon$ for all $n \geq k_1$ and $1 - \varepsilon < a^{-\frac{1}{n}} < 1 + \varepsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $1 - \varepsilon < a^{\frac{1}{k}} < 1 + \varepsilon$ and $1 - \varepsilon < a^{-\frac{1}{k}} < 1 + \varepsilon$.

Since $\lim_{n\to\infty} x_n = 0$, there exists a natural number p

Such that $-\frac{1}{k} < x_n < \frac{1}{k}$ for all $n \geq p$.

Let $a > 1$. Then $a^{-\frac{1}{k}} < a^{x_n} < a^{\frac{1}{k}}$ for all $n \geq p$

Or, $1 - \varepsilon < a^{-\frac{1}{k}} < a^{x_n} < a^{\frac{1}{k}} < 1 + \varepsilon$ for all $n \geq p$

Let $0 < a < 1$. Then $a^{\frac{1}{k}} < a^{x_n} < a^{-\frac{1}{k}}$ for all $n \geq p$.

Or, $1 - \varepsilon < a^{\frac{1}{k}} < a^{x_n} < a^{-\frac{1}{k}} < 1 + \varepsilon$ for all $n \geq p$

Therefore if $a > 0$, $1 - \varepsilon < a^{x_n} < 1 + \varepsilon$ for all $n \geq p$

This implies that $\lim_{n\to\infty} a^{x_n} = 1$.

Corollary1: If $\lim_{n\to\infty} x_n = l$ and $a > 0$, then $\lim_{n\to\infty} a^{x_n} = l$.

Corollary2: If $\lim_{n\to\infty} x_n = l$, then $\lim_{n\to\infty} e^{x_n} = l$.

Ex 7: If $u_n > 0$ and $\lim_{n\to\infty} u_n = u > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} v_n = v$, then $\lim_{n\to\infty} (u_n)^{v_n} = u^v$.

(H.W.)

Theorem: A convergent sequence is bounded.

CU'2008

Proof: Let $\{u_n\}$ be a convergent sequence and let l be its limit. Let us choose $\varepsilon = 1$. For this chosen ε there exists a natural number k

Such that $l-1 < u_n < l+1$ for all $n \geq k$.

Let $B = \max\{u_1, u_2, \ldots\ldots\ldots\ldots, u_{k-1}, l+1\}$ and $b = \min\{u_1, u_2, \ldots\ldots\ldots\ldots, u_{k-1}, l-1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$

This proves that the sequence $\{u_n\}$ is a bounded sequence.

Corollary: An unbounded sequence is not convergent.

Note (CU'2008): A bounded sequence may not be a convergent sequence. For example, the sequence $\{(-1)^n\}$ is a bounded sequence but the sequence does not converge to a limit.

Theorem: Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that converge to u and v respectively.

Then (i) $\lim_{n\to\infty}(u_n + v_n) = u + v$;

(ii) if $c \in \mathbb{R}$, $\lim_{n\to\infty}(cu_n) = cu$;

(iii) $\lim_{n\to\infty} u_n v_n = uv$;

(iv) $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{u}{v}$; provided $\{v_n\}$ is a sequence of non zero real numbers and $v \neq 0$.

Theorem: Let $\{u_n\}$ be a convergent sequence of real numbers converging to u. Then the sequence $\{|u_n|\}$ converges to $|u|$.

CU'1998

Proof: We have $||u_n| - |u|| \leq |u_n - u|$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} u_n = u$, there exists a natural number k

Such that $|u_n - u| < \varepsilon$ for all $n \geq k$.

It follows that for all $n \geq k$.

Since ε is arbitrary, $\lim_{n\to\infty} |u_n| = |u|$.

Note 1 (CU'1998): The converse of the theorem is not true. That is, if $\{|u_n|\}$ is a convergent sequence it does not necessarily imply that $\{u_n\}$ is a convergent sequence.

For example, let $u_n = (-1)^n$. Then the sequence $\{|u_n|\}$ converges to 1 but the sequence $\{u_n\}$ is not a convergent sequence.

Theorem: Let $\{u_n\}$ be a convergent sequence of real numbers and there exists a natural number m such that $u_n > 0$ for all $n \geq m$. Then $\lim_{n\to\infty} u_n \geq 0$