E - LEARNING MATERIALS

SEM - 4, CC - 8, Unit - 5

Topic - Power Series

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:POWER SERIES:-

Definition:-

A Series of the form $a_0 + a_1 x + a_2 x^2 + \cdots$, where $a_0, a_1, a_2, \ldots \in R$ is called a power series.

The general form of a power series is
$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$, where $a_0, a_1, a_2, \ldots \in R$ and $x_0 \in R$

This power series reduces to the standard form by the substitution $x - x_0 = x'$.

The power series

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges for all real x.

It is said to be everywhere convergent.

The power series

$1 + x + \lfloor 2\, x^2 + \lfloor 3\, x^3 + \cdots$ converges only for $x = 0$.

It is said to be nowhere convergent

The power series

$1 + x + x^2 + x^3 + \cdots$ converges for some real x and diverges for others. This is said to be neither nowhere convergent nor everywhere convergent.

We shall find the region of convergence of such a power series.

Th^m:- CH-'95'93

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1$, then the series is absolutely convergent.

Proof:- The series $a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + \cdots$ is convergent.

$\therefore \lim a_n x_1^n = 0$

Therefore the sequence $\{a_n x_1^n\}$ is bounded.

$\therefore |a_n x_1^n| < K$ for some real +ve number K [illegible]

Now $|a_n x^n| = |a_n x_1^n| \left|\left(\frac{x}{x_1}\right)^n\right|$

$\sum \left|\frac{x}{x_1}\right|^n$ is Convergent if $\left|\frac{x}{x_1}\right| < 1$

i.e if $|x| < |x_1|$

[illegible] $\left|\frac{x}{x_1}\right|^n = |a_n|^n$ is geometric series.

we have, $|a_n x^n| < K \left|\frac{x}{x_1}\right|^n$

unknown type

The series $\sum \left|\frac{x}{x_1}\right|^n$ is convergent for all x satisfying $|x| < |x_1|$.

By comparison test the series $\sum |a_n x^n|$ is convergent for all x satisfying $|x| < |x_1|$

i.e. the series $\sum a_n x^n$ is absolutely convergent for all real x satisfying $|x| < |x_1|$.

II Thm :- CH-'95

If the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = x_1$. Then the series is divergent for all real x satisfying $|x| > |x_1|$

[illegible] series [illegible] diverge [illegible]

$\Rightarrow$

Proof by Contradiction :

Let $c \in \mathbb{R}$ such that $|c| > |x_1|$

Let us assume that the series converges for $x = c$.

Since $|x_1| < |c|$, it follows from the previous theorem that x_1 is a point of convergence, a Contradiction.

This proves that the series is divergent for all x satisfying $|x| > |x_1|$.

CH-'98
CH-'97

Fundamental theorem of power series :-

$\sum_{n=0}^{\infty} a_n x^n$ be a power series which neither nowhere convergent nor everywhere convergent. Then there exist a +ve real number R such that the series converges absolutely for all x satisfying $|x| < R$ and the series diverges for all x satisfying $|x| > R$.

CH-98
[illegible] point of convergence [illegible] [illegible] point [illegible] converges [illegible] divergence [illegible]

Proof:

Since the series is neither nowhere convergent nor everywhere convergent, there exist at least one non-zero point of convergence, say $x = c$ and there exist at least one non-zero point of divergence, say $x = d$

Let $c_1 > 0$ be such that $c_1 < |c|$ and $d_1 > 0$ be such that $d_1 > |d|$. Then c_1 is a point of convergence and d_1 is a point of divergence of the series, by the previous theorem.

We assert that $c_1 < d_1$. Because if $c_1 > d_1$ then c_1 being a point of convergence of the series, d_1 will also be a point of convergence, a contradic

Let I_1 be the closed and bounded interval $[c_1, d_1]$. Then the series converges at c_1 and diverges at d_1.

Let $c_1' = \frac{1}{2}(c_1 + d_1)$. If c_1' be a point of convergence of the series we select the closed subinterval $[c_1', d_1]$ and call it $[c_2, d_2]$. If c_1' be a point of divergence of the series we select the closed subinterval $[c_1, c_1']$ and call it $[c_2, d_2]$.

Thus the closed sub interval $I_2 = [c_2, d_2]$ is such that -

(i) c_2 is a point of convergence and d_2 is a point of divergence of the series

(ii) $I_2 \subset I_1$

(iii) $|I_2| = \frac{1}{2}(d_1 - c_1)$

Let $c_2' = \frac{1}{2}(c_2 + d_2)$. If c_2' be a point of convergence of the series we select the closed subinterval $[c_2', d_2]$ and call it $[c_3, d_3]$.

If c_2' be a point of divergence of the series we select the closed sub interval $[c_2, c_2']$ and call it $[c_3, d_3]$.

Thus the closed subinterval $I_3 = [c_3, d_3]$ is such that

i) c_3 is a point of convergence and d_3 is a point of divergence of the series.

(ii) $I_3 \subset I_2$

(iii) $|I_3| = \frac{1}{2^2}(d_1 - c_1)$

Let $c_3' = \frac{1}{2}(c_3 + d_3)$.

Proceeding in the similar manner we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that for every $n \in \mathbb{N}$,

i) c_n is a point of convergence and d_n is a point of divergence of the series

ii) $I_{n+1} \subset I_n$

(iii) $|I_n| = \frac{1}{2^{n-1}}(d_1 - c_1)$

The sequence $\{I_n\}$ is a sequence of nested intervals and $\lim |I_n| = 0$.

By Cantor's theorem there exists one and only one point α such that

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$c_n \leq \alpha \leq d_n$ for all n and $\sup\{c_n\} = \alpha = \inf\{d_n\}$.

Let x_0 be such that $0 < x_0 < \alpha$

Since $\alpha = \sup\{c_n\}$, there exist a natural number m such that $x_0 < c_m \leq \alpha$

Since the power series converges at c_m and $0 < x_0 < c_m$, the power series converges for $x = x_0$.

By theorem I, the power series converges absolutely for all x such that $|x| < x_0$.

Since x_0 is arbitrary, the power series converges for all x satisfying $|x| < \alpha$.

Let x_1 be such that $x_1 > \alpha$.

Since $\alpha = \inf\{d_n\}$, there exist a natural number k such that $\alpha \leq d_k < x_1$

Since the power series is divergent for $x = d_k$ and $0 < d_k < x_1$, the power series diverges for $x = x_1$.

By theorem II, the power series diverges for all x satisfying $|x| \geq x_1$.

Since x_1 is arbitrary, the power series diverges for all x satisfying $|x| > \alpha$.

Hence $\alpha = R$ and the theorem is proved.

References.

1. Real Analysis - S.K. Mape.
2. Real Analysis - Malik & Arora.