E-Learning Materials

Sem-2, CC-03, Unit-1

Topic - Real Analysis. (Sets in $\mathbb{R}$)

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From (i) and (ii) it follows that $(A \cup B)' = A' \cup B'$ (1)
This completes the proof.

closed set: Let S be a subset of $\mathbb{R}$. S is said to be a closed set if $S' \subset S$ (i.e. if S contains all its limit points)

Note: Let S be a subset of $\mathbb{R}$. The derived set of S is a closed set.

24. The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$. Let $F = F_1 \cup F_2 \cup \ldots \cup F_m$.

Since F_i is a closed set $F_i' \subset F$ for $i = 1, 2, \ldots, m$

$$F' = (F_1 \cup F_2 \cup \ldots \cup F_m)' = F_1' \cup F_2' \cup \ldots \cup F_m'$$

$$F_1' \subset F_1 \Rightarrow F_1' \subset F, \quad F_2' \subset F_2 \Rightarrow F_2' \subset F, \ \ldots, \ F_m' \subset F_m \Rightarrow F_m' \subset F$$

Therefore $F_1' \cup F_2' \cup \ldots \cup F_m' \subset F$ i.e. $F' \subset F$.

As $F' \subset F$, F is a closed set and the theorem is done.

25. The intersection of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and let $F = F_1 \cap F_2 \cap \ldots \cap F_m$

Since F_i is a closed set, $F_i' \subset F_i$ for $i = 1, 2, \ldots, m$

$$F' = (F_1 \cap F_2 \cap \ldots \cap F_m)' \subset F_1' \cap F_2' \cap \ldots \cap F_m'$$

$$F_1' \cap F_2' \cap \ldots \cap F_m' \subset F_1' \subset F_1$$

$$F_1' \cap F_2' \cap \ldots \cap F_m' \subset F_2' \subset F_2$$

$$\ldots\ldots\ldots\ldots$$

$$F_1' \cap F_2' \cap \ldots \cap F_m' \subset F_m' \subset F_m$$

Therefore $F_1' \cap F_2' \cap \ldots \cap F_m' \subset F_1 \cap F_2 \cap \ldots \cap F_m = F$

It follows that $F' \subset F_1' \cap F_2' \cap \ldots \cap F_m' \subset F$

As $F' \subset F$, F is a closed set and the theorem is done.

26 The union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

Let us consider the sets F_i, where

$$F_1 = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$$

$$F_3 = \{x \in \mathbb{R} : -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}\}$$

$$\ldots\ldots\ldots$$

$$F_n = \{x \in \mathbb{R} : -\tfrac{1}{n} \leq x \leq \tfrac{1}{n}\}$$

$$\ldots\ldots\ldots$$

Each F_i is a closed set. $\bigcup_{i=1}^{\infty} F_i = F_1$ and this is a closed set.

Let us consider the sets F_i, where

$$F_1 = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$$

$$F_2 = \{x \in \mathbb{R} : \tfrac{1}{2} \leq x \leq 3 - \tfrac{1}{2}\}$$

$$\ldots\ldots\ldots$$

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$F_n = \{x \in \mathbb{R} : \frac{1}{n} \leq x \leq 3 - \frac{1}{n}\}$

Each F_i is a closed $\bigcup_{i=1}^{\infty} F_i = \{x \in \mathbb{R} : 0 < x < 3\}$. This is not a closed set.

25. The intersection of an arbitrary collection of closed set in $\mathbb{R}$ is a closed set.

let $\{F_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be a collection of closed sets in $\mathbb{R}$.

Then $(F_\alpha)' \subset F_\alpha$ for each $\alpha \in \Lambda$. let $F = \bigcap_{\alpha \in \Lambda} F_\alpha$

Case-1 $F' = \phi$ Then obviously $F' \subset F$

Case-2 $F' \neq \phi$. let $p \in F'$. Then p is a limit point of F.

let us choose $\varepsilon > 0$. Then $N'(p, \varepsilon)$ contains a point, say q, of F.

$q \in N'(p, \varepsilon) \cap F \Rightarrow q \in N'(p, \varepsilon) \cap F_\alpha$ for each $\alpha \in \Lambda$

This implies p is a limit of F_α for each $\alpha \in \Lambda$

Since each F_α is a closed set, $p \in F_\alpha$ for each $\alpha \in \Lambda$

Hence $p \in \bigcap_{\alpha \in \Lambda} F_\alpha$, i.e, $p \in F$

Thus $p \in F' \Rightarrow p \in F$ and therefore $F' \subset F$.

This proves that F is a closed set and the theorem is done.

26. let G be an open set in $\mathbb{R}$. Then the complement of G (in $\mathbb{R}$) is a closed set in $\mathbb{R}$.

Case-1: $G = \phi$ (an open set in $\mathbb{R}$). The complement of ϕ in $\mathbb{R}$ is $\mathbb{R}$ and $\mathbb{R}$ is a closed set.

Case-2: $G \neq \phi$. let $x \in G$. Since G is an open set, x is an interior point of G. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G$.

That is, $N(x) \cap G^c = \phi$ where G^c is the complement of G.

This implies that x is not a limit point of G^c. That is, $x \notin (G^c)'$

Thus $x \in G \Rightarrow x \notin (G^c)'$

contrapossitively, $x \in (G^c)' \Rightarrow x \notin G$ i.e. $x \in G^c$

Therefore $(G^c)' \subset G^c$ and this proves that G^c is a closed set.

This completes the proof.

27. let F be a closed set in $\mathbb{R}$. Then the complement of F (in $\mathbb{R}$) is an open set in $\mathbb{R}$.

Case-1: $F = \mathbb{R}$ (a closed set). Then the complement of F in $\mathbb{R}$ is ϕ and ϕ is an open set.

Case-2: F is a proper subset of $\mathbb{R}$. Then $F^c \neq \phi$ where F^c is the complement of F.

let $x \in F^c$. Since F is a closed set and $x \notin F$, x is not a limit point of F. Therefore there exists a neighbourhood $N(x)$ of x such that $N'(x) \cap F = \phi$.

Since $x \notin F$, $N(x) \cap F = \phi$. That is $N(x) \subset F^c$

This shows that x is an interior point of F^c. Since x is arbitrary F^c is an open set and the theorem is done.

27. Let S be a non empty subset of $\mathbb{R}$. bounded below and $m = \inf S$. If $m \notin S$ prove that m is limit point of S.

Let $\varepsilon > 0$. Since $m = \inf S$. we have

i) $x \in S \Rightarrow x > m$ [Since $m \notin S$]

ii) There exists an element y in S such that $m < y < m+\varepsilon$

or, $m-\varepsilon < m < y < m+\varepsilon$

or, $m-\varepsilon < y < m+\varepsilon$

or, $y \in (m-\varepsilon, m+\varepsilon)$

This shows that the ε-neighbourhood of m contains an element y in S.

Since $\varepsilon > 0$ is arbitrary, every neighbourhood of m contains infinitely many element of S.

Therefore m is a limit point of S.

28. Let G be a subset of $\mathbb{R}$ and G is open. Let F be a subset of $\mathbb{R}$ and F is closed. prove that $G-F$ is open and $F-G$ is closed.

$G-F = G \cap F^c$

Since F is closed set. So, The complement of F is open.

G and F^c are both open set. Then the intersection of two open set is open.

Therefore $G-F$ is open.

$F-G = F \cap G^c$

Since G is an open set. So, The complement of G is closed.

F and G^c are both closed. Then the intersection of two closed set is closed.

Therefore $F-G$ is closed.

29. If G be an open set and S be a finite subset of G. prove that $G-S$ is an open set.

$G-S = G \cap S^c$

S be a finite subset of G. So, S is closed. The complement of S is open.

we know that the intersection of two open set is an open set.