E - LEARNING MATERIALS.

SEM - 2 , CC - 03.

UNIT - 1.

Topic - Real Analysis

Sub Topic - Real Number System.

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1. Well ordering property: Every non empty subset of $\mathbb{N}$ has a least element.

2. principle of Induction: Let S be a subset of $\mathbb{N}$ such that
   i) $1 \in S$ and
   ii) if $k \in S$, then $k+1 \in S$

   Then $S = \mathbb{N}$

   proof: Let $T = \mathbb{N} - S$. We prove that $T = \emptyset$

   Let T be non empty. Then by the well ordering property of $\mathbb{N}$, the non-empty subset T has a least element, say m.

   Since $1 \in S$ and 1 is the least element of $\mathbb{N}$, $m > 1$.

   Hence $m-1$ is a natural number and $m-1 \notin T$. So, $m-1 \in S$

   by (ii) $m-1 \in S \Rightarrow (m-1)+1 \in S$, i.e $m \in S$

   This contradicts that m is the least element in T. Therefore our assumption is wrong and $T = \emptyset$.

   Therefore $S = \mathbb{N}$. This completes the proof.

3. Density property of $\mathbb{Q}$: If x and y be any two rational numbers and $x < y$, there exists a rational number r such that $x < r < y$. That is, between any two rational numbers there exists a rational number.

4. prove that $\sqrt{2}$ is irrational number.

   Let us assume that $\sqrt{2}$ is rational.

   Let $\sqrt{2} = \frac{p}{q}$ [where $q \neq 0$, p and $q \in \mathbb{Z}$, p and q are prime to each other]

   So, $2 = \frac{p^2}{q^2}$

   $\Rightarrow p^2 = 2q^2$

   $\Rightarrow p^2$ is even

   $\Rightarrow p$ is even

   Let $p = 2m$ for some $m \in \mathbb{Z}$

   $2q^2 = 4m^2$

   $\Rightarrow q^2 = 2m^2$

   $\Rightarrow q^2$ is even

   $\Rightarrow q$ is even

   Thus p and q are both even and this contradicts the assumption

Therefore $\sqrt{2}$ is irrational number.

5. Completeness property of $\mathbb{R}$:

upper bound: Let S be a subset of $\mathbb{R}$. A real number u is said to be an upper bound of S if $x \in S \Rightarrow x \leq u$.

lower bound: Let S be a subset of $\mathbb{R}$. A real number l is said to be a upper bound of S if $x \in S \Rightarrow x \geq l$.

bounded above: Let S be a subset of $\mathbb{R}$. S is said to be bounded above if S has an upper bound.

bounded below: Let S be a subset of $\mathbb{R}$. S is said to be bounded below if S has a lower bound.

bounded: Let S be a subset of $\mathbb{R}$. S is said to be a bounded set if S be bounded above as well as bounded below.

6. Give an example of bounded set.

Let $S = \{x \in \mathbb{R} : 1 < x < 2\}$, S is bounded set as 2 is an upper bound and 1 is a lower bound of S.

7. Supremum (least upper bound): Let S be a subset of $\mathbb{R}$. If S be bounded above, then an upper bound of S is said to be the supremum of S if it is less than every other upper bound of S.

Infimum (greatest lower bound): Let S be a subset of $\mathbb{R}$. If S be bounded below, then a lower bound of S is said to be the infimum of S if it is greater than every other lower bound of S.

8. Statement of completeness property:

Supremum property: Every non empty subset of $\mathbb{R}$ that is bounded above, has a least upper bound (or a supremum)

we assume the supremum property of $\mathbb{R}$ as an axiom.

Infimum property: Every non empty subset of $\mathbb{R}$. that is bounded below, has a greatest lower bound.

we assume the infimum property of $\mathbb{R}$ ...

9. let S be a non empty subset of $\mathbb{R}$, bounded below. Then S has an infimum.

let l_0 be a lower bound of S. let $T = \{l \in \mathbb{R} : l$ is a lower bound of $S\}$. Then T is a non empty subset of $\mathbb{R}$ because $l_0 \in T$.

Moreover $x \in T$ and $s \in S \Rightarrow x \leq s$. This shows that T is bounded above.

Thus T is non empty subset of $\mathbb{R}$, bounded above. By the supremum property of $\mathbb{R}$, T has a supremum. Let $\sup T = L$.

Then i) $l \leq L$ for every $l \in T$. Since L is an upper bound of T.

and ii) Since every $s \in S$ is an upper bound of T and $L = \sup T$, $L \leq s$ for every $s \in S$.

(ii) shows that L is a lower bound of S and (i) shows that $L \geq$ any lower bound of S. consequently $L = \inf S$.

Therefore S has an infimum and the proof is complete.

10. Complete ordered field:

An ordered field is said to be a complete ordered field if the completeness property holds in it.

11. properties of the supremum and the infimum:

let S be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup S$ exists. Let $M = \sup S$. Then $M \in \mathbb{R}$ and M satisfies the following conditions →

i) $x \in S \Rightarrow x \leq M$ and

ii) for each $\epsilon > 0$, there exists an element $y(\epsilon)$ in S such that $M - \epsilon < y \leq M$.

let S be a non-empty subset of $\mathbb{R}$, bounded below. Then $\inf S$ exists. Let $m = \inf S$. Then $m \in \mathbb{R}$ and m satisfies the following conditions →

i) $x \in S \Rightarrow x \geq m$ and

ii) for each $\epsilon > 0$, there exists an element $y(\epsilon)$ in S such that $m \leq y < m + \epsilon$.

12. prove that the set $\mathbb{N}$ is not bounded set. (above) $\boxed{2011}$

The set $\mathbb{N}$ is a non empty subset of $\mathbb{R}$, Since $1 \in \mathbb{N}$

let $\mathbb{N}$ be bounded above. Then $\mathbb{N}$ being a non empty subset of $\mathbb{R}$, bounded above, $\sup N$ exists by the Supremum property of $\mathbb{R}$. Let $u = \sup N$. Then i) $x \in N \Rightarrow x \le u$ and (4)

ii) for each $\varepsilon > 0$ there exists an element, say y, in N such that $u - \varepsilon < y \le u$

Let us choose $\varepsilon = 1$. Then there exists an element k in N such that $u - 1 < k \le u$.

$\Rightarrow u - 1 + 1 < k + 1 \le u + 1$

$\Rightarrow u < k + 1 \le u + 1$

$\Rightarrow u < k + 1$

Since k is a natural number, $k+1$ is also a natural number. $k + 1 > u$ implies that u is not an upper bound of N.

Thus we arrive at a contradiction. So our assumption that N is bounded above is wrong. Hence the set N is not bounded above.

13. Let S be a non empty subset of $\mathbb{R}$, bounded above and $T = \{-x : x \in S\}$. prove that the set T is bounded below and $\inf T = -\sup S$.

$\sup S$ exists. Let $u = \sup S$. Then $x \in S \Rightarrow x \le u$

Let $y \in T$. Then $-y \in S$ and therefore $-y \le u$ i.e $y \ge -u$. This implies that $-u$ is a lower bound of T. Therefore the set T is bounded below.

Let us choose $\varepsilon > 0$. Since $u = \sup S$, there exists an element p in S such that $u - \varepsilon < p \le u$. Therefore $-u \le -p < -u + \varepsilon$ — (1)

Let $q = -p$. Then $q \in T$

(1) shows that for positive ε there exists an element q in T such that $-u \le q < -u + \varepsilon$

This proves that $-u = \inf T$. Therefore $\inf T = -\sup S$

14. Let A, B be bounded subsets of $\mathbb{R}$ such that $x \in A, y \in B \Rightarrow x \le y$ prove that $\sup A \le \inf B$.

Since A, B are non-empty bounded subsets of $\mathbb{R}$, $\sup A$ and $\inf B$ exist. Let $\sup A = a^*$, $\inf B = b_*$

Let $b \in B$. Then $x \in A \Rightarrow x \le b$. This shows that b is an upper bound of A. Since $\sup A = a^*$ and b is an upper bound of A it follows that $a^* \leq b$.

Now $a^* \leq b$ for all $b \in B$. Therefore a^* is a lower bound of B. Since $\inf B = b_*$ and a^* is a lower bound of B it follows that $a^* \leq b_*$, i.e. $\sup A \leq \inf B$.

15. Any finite set is bounded.

let $S = \{x_1 \ldots\ldots x_n\}$ be any finite set.

let $M = \max\{x_1 \ldots\ldots x_n\}$, $m_1 = \min\{x_1 \ldots\ldots x_n\}$

$x_i \leq M$, $x_i \geq m$ $i = 1, \ldots\ldots, n$

Here M is the upperbound of S.

and m is the lower bound of S.

Therefore S is bounded.

So, we can say that any finite set is bounded.

16. $0 \leq |a-b| < \varepsilon$ prove that $a = b$.

we assume that $a > b$

$$a - b > 0$$

$$\frac{1}{2}(a-b) > 0$$

let $\varepsilon = \frac{1}{2}(a-b)$

by the given condition

$$0 \leq |a-b| < \varepsilon$$

$$\Rightarrow 0 \leq a-b < \varepsilon$$

$$\Rightarrow 0 \leq a-b < \frac{1}{2}(a-b)$$

$\therefore$ $a-b < \frac{1}{2}(a-b)$ [a contradiction]

Therefore $a = b$

17. Archimedean property of $\mathbb{R}$: 2011 2013

If $x, y \in \mathbb{R}$ and $x > 0$, $y > 0$, then there exists a natural number n such that $ny > x$.

<u>proof:</u> If possible, let there exist no natural number n for which $ny > x$. Then for every natural number k, $ky \leq x$

Thus the set $S = \{ky : k \in \mathbb{N}\}$ is bounded below, x being an upper bound. S is non empty because YES.

By the Supremum property of $\mathbb{R}$, $\sup S$ exists. let $\sup S = M$

(i) $KY \leq M \ \forall \ KY \in S$

(ii) $M - \varepsilon < my \leq M$

let $\varepsilon = Y$, Since $Y > 0$

$M - y < py \Rightarrow M < py + y$

$M < (p+1)y \qquad p+1 \in N$

$M < qy$

This shows that M is not the supremum of S, a contradiction.

Therefore our assumption is wrong and the existence of a natural number n satisfying $ny > x$ is proved.

18. If $x \in \mathbb{R}$. then there exists a natural number n such that $n > x$.

Case-1: $x > 0$

Taking $y = 1$. by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$ and hence the existence is proved.

Case-2: $x \leq 0$. Then $n = 1$

19. If $x \in \mathbb{R}$ and $x > 0$, then there exists a natural number n such that $0 < \frac{1}{n} < x$.

Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $nx > 1$

Since n is a natural number, $n > 0$ and therefore $\frac{1}{n} > 0$ and $x > \frac{1}{n}$.

Therefore we have $0 < \frac{1}{n} < x$.

20. If $x \in \mathbb{R}$ and $x > 0$, there exists a natural number m such that $m - 1 \leq x < m$

Taking $y = 1$ and $x > 0$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$, i.e $n > x$

let $S = \{K \in N : K > x\}$. Then S is non empty subset of N, Since $n \in S$. By the well ordering property of the set N, S has a least element, say m. Since $m \in S$, $m > x$.

As m is the least element in S, $m - 1 \not> x$ i.e $m - 1 \leq x$

Hence $m - 1 \leq x < m$.

21. If $x \in \mathbb{R}$, then there exists an integer m such that $m-1 \leq x < m$. (7)

Case-1: $x > 0$

This is 20.

case-2: $x = 0$

In this case $m = 1$

case-3: $x < 0$

First we assume that x is not a negative integer.

Then $-x > 0$. By case 1. there exists a natural number m' such that $m'-1 < -x < m'$

$-x < m' \Rightarrow x > -m'$ and $m'-1 < -x \Rightarrow x < -m'+1$

Therefore $-m' < x < -m'+1$

let $m = -m'+1$. Since m' is a natural number, m is an integer ≤ 0

So, we have $m-1 < x < m$

If however x is a negative integer, then $x = m-1$

combining, we have $m-1 \leq x < m$

22. Density property of $\mathbb{R}$. 2012

1. If x, y are real numbers with $x < y$. then there exists a rational number r such that $x < r < y$.

2. If x, y are real numbers with $x < y$, then there exists a irrational numbers/r' such that $x < s/r' < y$

23. Find $\sup A$ and $\inf A$, where

$A = \{x \in \mathbb{R} : 3x^2 + 8x - 3 < 0\}$ 2010

$= \{x \in \mathbb{R} : 3x(x+3) - (x+3) < 0\}$

$= \{x \in \mathbb{R} : (x+3)(3x-1) < 0\}$

when $(x+3) < 0$ then $(3x-1) > 0$

$x < -3$ and $x > \frac{1}{3}$

This is impossible.

when $(x+3) > 0$ Then $(3x-1) < 0$

$x > -3$ $x < \frac{1}{3}$

combining these two we get $-3 < x < \frac{1}{3}$

Therefore $A = \{x \in \mathbb{R} : -3 < x < \frac{1}{3}\}$

Therefore $\sup A = \frac{1}{3}$ and $\inf A = -3$.

ii) $A = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in N \right\}$

$= \left\{ 0 < \frac{1}{m} + \frac{1}{n} \leq 2 ,\ m, n \in N \right\}$

$\sup A = 2$ and $\inf A = 0$

iii) $A = \left\{ \frac{n + (-1)^n}{n} : n \in N \right\}$

$= \left\{ 0 \leq \frac{n+(-1)^n}{n} \leq \frac{3}{2} ,\ n \in N \right\}$

$\sup A = \frac{3}{2}$ and $\inf A = 0$

24. Let A and B be two non empty bounded subsets of $\mathbb{R}$. prove that the set $c = \{x + y : x \in A, y \in B\}$ is bounded. prove that $\sup c = \sup A + \sup B$.

Since A is bounded set $\sup A$ and $\inf A$ both exists.
let $a = \sup A$, $\inf A = a'$

Since B is bounded set, so $\sup B$ and $\inf B$ both exists.
let $b = \sup B$ and $\inf B = b'$

Now, $x \in A \Rightarrow a' \leq x \leq a$

$y \in B \Rightarrow b' \leq y \leq b$

That is $a' + b' \leq x + y \leq a + b \ \forall\ x \in A$ and $y \in B$

This shows that $a+b$ is an upper bound of c and $a'+b'$ is a lower bound of c.

i.e c is a bounded set.

Here $\sup c$ exists.

and $\sup c \leq a + b$

we show that $\sup c = a + b$

If possible let $p < a+b$ be an upper bound of c.

let $a + b - p = 2\varepsilon$

$a + b - \varepsilon = p + \varepsilon$

Since $\sup A = a$ there exists an element a_1 in A s.t $a - \frac{\varepsilon}{2} < a_1 \leq a$

Since $\sup B = b$, there exists an element b_1 in B s.t $b - \frac{\varepsilon}{2} < b_1 \leq b$

Therefore, $a + b - \varepsilon < a_1 + b_1 \leq a + b$

i.e $a_1+b_1 > a+b-\varepsilon$

i.e $a_1+b_1 > p+\varepsilon$

Thus p fails to an upper bound of c.

Hence such p does not exists.

i.e $a+b$ is the least upper bound of c.

i.e $\operatorname{Sup} c = a+b$

i.e $\operatorname{Sup} c = \operatorname{Sup} A + \operatorname{Sup} B$

25. Show that $\inf A = -\operatorname{Sup}(-A)$ where A is a non empty set of $\mathbb{R}$. bounded below and $-A = \{-x : x \in A\}$

A is non empty and bounded below. Therefore $\inf A$ exists.

let $m = \inf A$

Then $m \leq x \;\forall\; x \in A$ —①

Also for each $\varepsilon > 0$ there exists an element $y \in A$ s.t

$$m \leq y < m+\varepsilon \quad —②$$

From ① we get →

$$-x \leq -m \quad \forall\; -x \in -A \quad —③$$

This shows that $-A$ is bounded above.

Therefore $\sup(-A)$ exists.

From ② we get

$$-y > -m-\varepsilon \;,\; -y \in -A \quad —④$$

From ③ and ④ we get →

ⓐ $-x \leq -m \quad \forall\; -x \in -A$

ⓑ $-m-\varepsilon < -y \leq -m \;,\; -y \in -A$

i.e, $-m = \operatorname{Sup}(-A)$

i.e, $m = -\operatorname{Sup}(-A)$

i.e, $\inf A = -\operatorname{Sup}(-A)$ [proved]

26. let S and T be not empty subsets of $\mathbb{R}$, such that $x \leq t \;\forall\, x \in S$ and $t \in T$. If T has a Supremum then show that S has a Supremum and $\sup S \leq \inf T \leq \sup T$

$\sup T$ exists.

Then $t \in T \Rightarrow t \leq M$

Therefore $x \leq t \leq M \ \forall x \in S$

i.e $x \leq M \ \forall x \in S$

Therefore S is bounded above.

i.e $\sup S$ exists.

let $k = \sup S$.

T is bounded below. as $x \leq t \ \forall t \in T$

so, $\inf T$ exists.

let $m = \inf T$

Then $k > m$ is false

That is $k \leq m$ holds.

i.e $k \leq m \leq M$

$\sup S \leq \inf T \leq \sup T$.