E-Learning Materials

Sem-2, CC-03, Unit-1

Topic - Real Analysis. (Sets in $\mathbb{R}$)

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① **Neighbourhood:** Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a neighbourhood of c if there exists an open interval (a, b) such that $c \in (a, b) \subset S$. ①

② **Interior point:** Let S be a subset of $\mathbb{R}$. A point x in S is said to be an interior point of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

③ **Open set:** Let $S \subset \mathbb{R}$. S is said to be an open set if each point of S is an interior point of S.

④ Let $S \subset \mathbb{R}$. Then S is an open set if and only if $S = \text{int} S$.

We prove that the theorem for a non empty set S because $S = \phi$ then $\phi = \text{int} S$ holds and ϕ is an open set.

Let S be a non empty open set and $x \in S$. Then x is an interior points of S.

Thus $x \in S \Rightarrow x \in \text{int} S$. Therefore $S \subset \text{int} S$ —— (i)

Let $y \in \text{int} S$. Then $y \in S$ by the defination of an interior point.

Thus $y \in \text{int} S \Rightarrow y \in S$. Therefore $\text{int} S \subset S$ —— (ii)

From (i) and (ii) we have $S = \text{int} S$.

Conversely, let S be a non-empty set and $S = \text{int} S$.

Let $x \in S$. Then $x \in \text{int} S$, since $S = \text{int} S$.

Thus each point of S is an interior point of S and therefore S is an open set.

This completes the proof.

⑤ The union of two open sets in $\mathbb{R}$ is an open set.

Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Let $x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$

Let $x \in G_1$. Since G_1 is open set and $x \in G_1$, x is an interior point of G_1. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_1$

$N(x) \subset G_1 \Rightarrow N(x) \subset G_1 \cup G_2$

This shows that x is an interior point of $G_1 \cup G_2$

Since x is arbitrary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is an open set.

If however $x \in G_2$, we can prove in a similar manner that $G_1 \cup G_2$ is an open set. This completes the proof.

⑥ The intersection of two open sets in $\mathbb{R}$ is an open set.

Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Case-I: $G_1 \cap G_2 = \phi$. Since ϕ is an open set, $G_1 \cap G_2$ is an open set.

Case-II: $G_1 \cap G_2 \neq \phi$. Let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$.
Since G_1 is an open set and $x \in G_1$, x is an interior point of G_1.
Hence there exists a possitive δ_1 such that the neighbourhood
$N(x, \delta_1) \subset G_1$

Since G_2 is an open set and and $x \in G_2$, x is an interior point of G_2.
Hence there exists a possitive δ_2 such that the neighbourhood
$N(x, \delta_2) \subset G_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$

$N(x, \delta) \subset N(x, \delta_1) \subset G_1$ and $N(x, \delta) \subset N(x, \delta_2) \subset G_2$

consequently $N(x, \delta) \subset G_1 \cap G_2$

This shows that x is an interior point of $G_1 \cap G_2$. Since x is arbitrary, $G_1 \cap G_2$ is an open set and this completes the proof.

⑦ The union of a finite number of open sets in $\mathbb{R}$ is an open set.

Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cup G_2 \cup \ldots \cup G_m$

Let $x \in G$. Then x belongs to at least one of the sets, say G_K.
Since G_K is an open set and $x \in G_K$, x is an interior point of G_K.
Hence there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_K$.

$N(x) \subset G_K \Rightarrow N(x) \subset G$

This shows that x is an interior point of G. Since x is arbitrary G is an open set. This completes the proof.

⑧ The intersection of a finite number of open set in $\mathbb{R}$ is an open set.

Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cap G_2 \cap \ldots \cap G_m$

Case 1: $G = \phi$. Then G is an open set, since ϕ is an open set.

Case 2: $G \neq \phi$. Let $x \in G$. Then $x \in G_i$ for each $i = 1, 2, \ldots m$.
Since G_1 is an open set and $x \in G_1$, there exists a possitive δ_1 such that $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2$, there exists a possitive δ_2 Such that $N(x, \delta_2) \subset G_2$

Since G_m is an open set and $x \in G_m$, there exists a possitive δ_m Such that $N(x, \delta_m) \subset G_m$

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$

$N(x, \delta) \subset N(x, \delta_1) \subset G_1$

$N(x, \delta) \subset N(x, \delta_2) \subset G_2$

$N(x, \delta) \subset N(x, \delta_m) \subset G_m$

consequently $N(x, \delta) \subset G_1 \cap G_2 \cap \cdots \cap G_m = G$

This show that x is an interior point of G. Since x is arbitrary, G is open set and the proof is complete.

9. The union of an arbitrary collection of open set in $\mathbb{R}$ is an open set.

Let $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be an arbitrary collection of open sets in $\mathbb{R}$. Let $G = \bigcup_{\alpha \in \Lambda} G_\alpha$

Let $x \in G$. Then x belongs to at least one open set of the collection Say G_λ $(\lambda \in \Lambda)$

Since G_λ is an open set and $x \in G_\lambda$, x is an interior point of G_λ.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_\lambda$.

$N(x) \subset G_\lambda \Rightarrow N(x) \subset G$

This show that x is an interior point of G. Since x is arbitrary, G is an open set and the proof is complete.

10. The intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Let us consider the sets G_i where

$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$

$G_2 = \{x \in \mathbb{R} : -\frac{1}{2} < x < \frac{1}{2}\}$

$G_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}\}$

Each G_i is an open set. $\bigcap_{i=1}^{\infty} G_i = \{0\}$. This is not an open set.

Let us consider the sets G_i where

$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$

$G_2 = \{x \in \mathbb{R} : -2 < x < 2\}$

$G_n = \{x \in \mathbb{R} : -n < x < n\}$

Each G_i is an open set. $\bigcap_{i=1}^{n} G_i = G_1$. This is an open set. (4)

From these two examples we conclude that the intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

11. Let S be a subset of $\mathbb{R}$. Then int$S$ is an open set.

<u>Case:1</u> int$S = \phi$. Since ϕ is an open set, int$S$ is an open set.

<u>case:2</u> int$S \neq \phi$. Let $x \in$ int$S$. Then x is an interior point of S. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

Let $y \in N(x)$. Then $N(x)$ is a neighbourhood of y also and since $N(x) \subset S$, y is an interior point of S.

Thus $y \in N(x) \Rightarrow y \in$ int$S$. Therefore $N(x) \subset$ int$S$.

This shows that x is an interior point of int$S$.

Thus $x \in$ int$S \Rightarrow x$ is an interior point of int$S$

Therefore int$S$ is an open set. This completes the proof.

12. Let $S \subset \mathbb{R}$. Then int$S$ is the largest open set contained in S.

By the previous theorem, int$S$ is an open set and int$S \subset S$, by defination.

Let P be any open set contained in S.

Let $x \in P$. Since P is an open set, x is an interior point of P. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset P$.

But $N(x) \subset P \Rightarrow N(x) \subset S$, since $P \subset S$.

This shows that x is an interior point of S, i.e, $x \in$ int$S$.

Thus $x \in P \Rightarrow x \in$ int$S$. Therefore $P \subset$ int$S$.

Since P is arbitrary, int$S$ is the largest open set contained in S.

13. <u>Limit point:</u> Let S be a subset of $\mathbb{R}$. A point p in $\mathbb{R}$ is said to be a limit point (or an accumulation point, or a cluster point) of S if every neighbourhood of p contains a point of S other than p.

$$[N(p,\varepsilon) - \{p\}] \cap S \neq \phi$$

<u>Note:</u> A limit point of S may or may not belong to S.

14. <u>Isolated point:</u> Let S be a subset of $\mathbb{R}$. A point x in S is said to be an isolated point of S if x is not a limit point of S.

$$N'(x) \cap S = \phi$$

<u>Theorem:</u> Let $S \subset \mathbb{R}$ and p be a limit point of S. Then every neighbourhood of p contains infinitely many elements of S.

References.

1. Real Analysis - S.K. Mape.

2. Real Analysis – Malik & Arora.