Definition 2.1.2 : Let $f : [a, b] \to \mathbb{R}$. f is said to be continuous almost everywhere ($a{\cdot}e$) in $[a, b]$ if the set of all discontinuities of f in $[a, b]$, is a set of measure zero.

i.e. if f is discontinuous at each point of $A \subset [a, b]$ and A is a set of measure, then f is continuous $a{\cdot}e$ in $[a, b]$.

Example 1.5 : Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0, f(x) = (-1)^n, \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, n = 0, 1, 2, ...$

Then f is discontinuous at the points $0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$

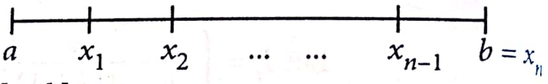
Since the set of discontinuities of f of $[0, 1]$ is of measure zero, f is continuous $a{\cdot}e$ in $[0, 1]$.

2.2 Partition

Definition 2.1 : Let $[a, b]$ be closed and bounded interval. By a partitions of $[a, b]$ we shall mean a finite set $P = \{x_0, x_1, x_2, ..., x_n\}$ of points in $[a, b]$ such that $a = x_0 < x_1 < x_2 < ... < x_n = b$.

A partition of $[a, b]$ may also be written as :

$$P = \{a = x_0 < x_1 < x_2 < ... < x_n = b\}$$



Let $P = \{a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of $[a, b]$.

Then P divides the interval $[a, b]$ into n non-overlapping subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$.

$P_1 = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{4}{5}, 1\right\}$ is a partition of $[0, 1]$

$P_2 = \left\{0, \frac{2}{7}, \frac{1}{2}, \frac{5}{7}, 1\right\}$ is a partition of $[0, 1]$

$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n} = 1\right\}$ is a partition of $[0, 1]$

$P = \{a, a + h, a + 2h, ..., a + (n - 1)h, a + nh\}$ is a partition of $[a, b]$

Norm of a partition :

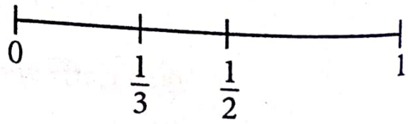
Definition 2.2.2 : Let $[a, b]$ be closed and bounded interval and $P = \{a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of $[a, b]$. Then norm (or mesh) of P denoted by $\| P \|$ (or $\mu(P)$), is defnined by

$$\| P \| = \max_{1 \le r \le n} | x_r - x_{r-1} |$$

i.e. $\| P \| :=$ the maximum length of the subintervals

Let $P = \left\{0, \frac{1}{3}, \frac{1}{2}, 1\right\}$ be a partition of $[0, 1]$.

Then $\| P \| = \max \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}$



$P = \{a, a + h, a + 2h, ..., a + nh = b\}$ be a partition of $[a, b]$.

$$\| P \| = h = \frac{b-a}{n}$$

Remark : For a given closed and bounded interval $[a, b]$, since any finite number of points in $[a, b]$ can be chosen in infinitely many ways, there are infinite number of partitions of $[a, b]$. The family of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

$\therefore P \in \mathcal{P}[a, b] \Rightarrow P$ is a partion of $[a, b]$

Refinement of a partition :

Definition 2.2.3 : Let $[a, b]$ be closed and bounded interval and P is a partitions of $[a, b]$. A partition Q of $[a, b]$ is said to be a refinement of P or Q is finer that P if $Q \supset P$.

$\therefore$ if every points of P is used in the construction of Q. i.e. if Q is obtained from P by adjoining some additional points to P, then Q is refinement of P.

$P = \left\{0, \frac{1}{3}, \frac{1}{2}, 1\right\}$ and $Q = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{4}{5}, 1\right\}$ are two partitions of $[0, 1]$. Since $P \subset Q$, Q is a refinement of P.

For any two partitions P_1, P_2 of $[a, b]$, $P^* = P_1 \cup P_2$ is a common refinement of P_1 and P_2.

Remark : Let $[a, b]$ be closed and bounded interval and P is a partition of $[a, b]$. If Q is any refinement of $[a, b]$, then $\|Q\| \le \|P\|$.

$\therefore$ norm of a refinement can not exceed the norm of a partition.

Lower and Upper Darboux sums :

Definition 2.2.4 : Let $f: [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P is a partition of $[a, b]$. Then f is bounded in $[x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad r = 1, 2, \ldots, n$

Then the sum $\sum\limits_{r=1}^{n} m_r (x_r - x_{r-1})$, denoted by $L(P, f)$, is called lower Darboux sum of f corresponding to the partition P of $[a, b]$ and the sum $\sum\limits_{r=1}^{n} M_r (x_r - x_{r-1})$, denoted by $U(P, f)$, is called upper Darboux sum of f corresponding to the partition P of $[a, b]$.

$$\therefore L(P, f) = \sum_{r=1}^{n} m_r (x_r - x_{r-1}) \text{ and } U(P, f) = \sum_{r=1}^{n} M_r (x_r - x_{r-1}).$$

Lower and upper integral :

Definition 2.2.5 : Let $f: [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then for every $P \in \mathcal{P}[a, b]$, we have two real numbers $L(P, f)$ and $U(P, f)$. The suprimum of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$, denoted by $\underline{\int_a^b} f$ or $\underline{\int_a^b} f(x)\, dx$, is called lower integral of f on $[a, b]$ and the infimum of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$, denoted by $\overline{\int_a^b} f$ or $\overline{\int_a^b} f(x)\, dx$, is called upper integral of f on $[a, b]$.

$$\therefore \underline{\int_a^b} f = \sup\{L(P, f) : P \in \mathcal{P}[a, b]\} \text{ and } \overline{\int_a^b} f = \inf\{U(P, f) : P \in \mathcal{P}[a, b]\}$$

Definition 2.2.6 : Let $f: [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then f is R-integrable on $[a, b]$ iff

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Remark : The class of all R-integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.

(3)

Example 2.2.1 : Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P is a partition of $[a, b]$. Then $m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$, where M and m are respectively the suprimum and infimum of f on $[a, b]$.

Solution : Let $P = \{a = x_0 < x_1 < x_2 < ... < x_n = b\}$.

Now f is bounded on $[a, b]$

$\Rightarrow f$ is bounded on $[x_{r-1}, x_r], \forall r = 1, 2, ..., n$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, ..., n$.

Then $m \le m_r \le M_r \le M, \forall r = 1, 2, ..., n$

$$\Rightarrow m(x_r - x_{r-1}) \le m_r(x_r - x_{r-1}) \le M_r(x_r - x_{r-1}) \le M(x_r - x_{r-1}), \forall r = 1, 2, ..., n$$

$$\Rightarrow \sum_{r=1}^{n} m(x_r - x_{r-1}) \le \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M(x_r - x_{r-1})$$

$$\Rightarrow m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$$

Remark: From the above example, both the sets $\{L(P, f) ; P \in P[a, b]\}$ and $\{U(P, f) : P \in P[a, b]\}$ are bounded sets is $\mathbb{R}$, $M(b-a)$ is an upper bound and $m(b-a)$ is a lower bound to both the sets.

$$m(b-a) \le \underline{\int_a^b} f = \sup\{L(P, f) : P \in \mathcal{P}[a, b]\} \le M(b-a)$$

and $m(a-b) \le \inf\{U(P, f) : P \in \mathcal{P}[a, b]\} = \overline{\int_a^b} f \le M(b-a)$

SOLVED PROBLEMS

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = k, x \in [a, b]$. Show that $f \in \mathcal{R}[a, b]$ and $\int_a^b f = k(b-a)$.

Solution : $f(x) = k, x \in [a, b]$

f is bouned on $[a, b]$

Let $P = \{a = x_0 < x_1 < x_2 < ... < x_n = b\}$ be a partition of $[a, b]$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, ..., n$

Then $M_r = m_r = k, r = 1, 2, ..., n$

$$\therefore U(P, f) = \sum_{r=1}^{n} M_r(x_r - x_{r-1}) = \sum_{r=1}^{n} k(x_r - x_{r-1}) = k(b-a)$$

$$L(P, f) = \sum_{r=1}^{n} m_r(x_r - x_{r-1}) = \sum_{r=1}^{n} k(x_r - x_{r-1}) = k(b-a)$$

Since $P \in \mathcal{P}[a, b]$ is arbitrary, $U(P, f) = L(P, f) = k(b-a), \forall P \in \mathcal{P}[a, b]$

$\therefore \{U(P,f) : P \in \mathcal{P}[a, b]\} = \{k(b-a)\}$ and $\{L(P,f) : P \in \mathcal{P}[a, b]\} = \{k(b-a)\}$

$$\therefore \quad \overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}[a, b]\} = \inf\{k(b-a)\} = k(b-a)$$

$$\text{and } \underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a, b]\} = \sup\{k(b-a)\} = k(b-a)$$

Since, $\underline{\int_a^b} f = \overline{\int_a^b} f$, $f \in \mathcal{R}[a, b]$ and $\int_a^b f = k(b-a)$

Problem 2. Let $f: [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ -1, & x \in [0,1] - \mathbb{Q} \end{cases}$. Show that $f \notin R[0, 1]$.

Solution : $f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ -1, & x \in [0,1] - \mathbb{Q} \end{cases}$

Then $|f(x)| = 1, \forall x \in [0, 1]$.

$\Rightarrow f$ is bounded on $[0, 1]$

Let $P = \{0 = x_0 < x_1 < x_2 < \ldots < x_n = 1\} \in \mathcal{P}[0, 1]$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$

Then $M_r = 1, m_r = -1 \; \forall r = 1, 2, \ldots, n$

$$U(P,f) = \sum_{r=1}^{n} M_r (x_r - x_{r-1}) = \sum_{r=1}^{n} 1(x_r - x_{r-1}) = 1(1-0) = 1$$

$$\text{and } L(P,f) = \sum_{r=1}^{n} m_r (x_r - x_{r-1}) = \sum_{r=1}^{n} -1(x_r - x_{r-1}) = -1(1-0) = -1$$

Since $P \in \mathcal{P}[0, 1]$ is arbitrary, $U(P,f) = 1$ and $L(P,f) = -1, \forall P \in \mathcal{P}[0, 1]$

$$\overline{\int_0^1} f = \inf\{U(P,f) : P \in \mathcal{P}[0, 1]\} = \inf\{1\} = 1$$

$$\text{and } \underline{\int_0^1} f = \sup\{L(P,f) : P \in \mathcal{P}[0, 1]\} = \sup\{-1\} = -1$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$, $f \notin \mathcal{R}[0, 1]$

Problem 3. Let $f: [0, 1] \to \mathbb{R}$ be $f(x) = x^2, x \in [0, 1]$.

Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n} = 1\right\}$ be a partition of $[0, 1]$.

Show that $\sup\{L(P_n, f) : n \in \mathbb{N}\} = \inf\{U(P_n, f) : n \in \mathbb{N}\}$

Deduce that $f \in \mathcal{R}[0, 1]$

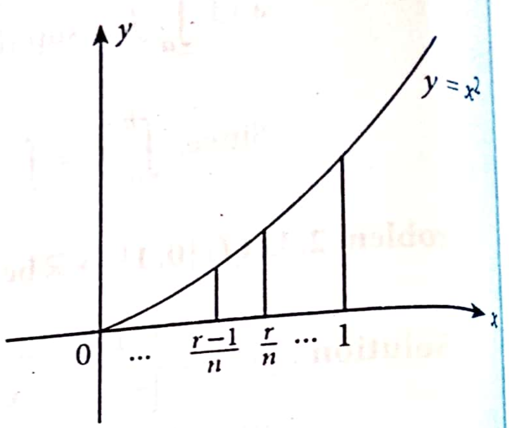
Solution : $f(x) = x^2, x \in [0, 1]$

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n} = 1\right\} \in P[0, 1]$$

Let $M_r = \sup\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $m_r = \inf\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $r = 1, 2, \ldots$

Since f is monotone increasing on $[0, 1]$, $M_r = f\left(\frac{r}{n}\right) = \frac{r^2}{n^2}$

and $m_r = f\left(\frac{r-1}{n}\right) = \frac{(r-1)^2}{n^2}$, $r = 1, 2, \ldots, n$



$$U(P_n, f) = \sum_{r=1}^{n} \frac{r^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3}\,(1^2 + 2^2 + \ldots + n^2)$$

$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$\inf\{U(P_n, f) : n \in \mathbb{N}\} = \lim_{n\to\infty} U(P_n f) = \frac{1}{6}\,(1+0)\,(2+0) = \frac{1}{3}$$

$$L(P_n, f) = \sum_{r=1}^{n} \frac{(r-1)^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right)$$

$$= \frac{1}{n^3}\,\frac{(n-1)\,n\,(2n-1)}{6}$$

$$= \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\sup\{L(P_n, f) : n \in \mathbb{N}\} = \lim_{n\to\infty} L(P_n, f) = \frac{1}{6}\,(1-0)\,(2-0) = \frac{1}{3}$$

$$\sup\{L(P_n, f) : n \in \mathbb{N}\} = \inf\{U(P_n, f) : n \in \mathbb{N}\} = \frac{1}{3} \quad \text{[Proved]}$$

Now $\{L(P_n, f) : n \in \mathbb{N}\} \subseteq \{L(P, f) : P \in \mathcal{P}[0, 1]\}$

$$\Rightarrow \frac{1}{3} = \sup\{L(P_n, f) : n \in \mathbb{N}\} \le \sup\{L(P, f) : P \in \mathcal{P}[0, 1]\} = \underline{\int_0^1} f \qquad \ldots (1)$$

Again, $\{U(P_n, f) : n \in \mathbb{N}\} \le \{U(P, f) : P \in \mathcal{P}[0, 1]\}$

$$\Rightarrow \overline{\int_0^1} f = \inf\{U(P, f) : P \in \mathcal{P}[0, 1]\} \le \inf\{U(P_n, f) : n \in \mathbb{N}\} = \frac{1}{3} \qquad \ldots (2)$$

Also, $\underline{\int_0^1} f \le \overline{\int_0^1} f \qquad \ldots (3)$

From (1), (2) and (3), $\frac{1}{3} \le \underline{\int_0^1} f \le \overline{\int_0^1} f \le \frac{1}{3}$

$$\Rightarrow \underline{\int_0^1} f = \overline{\int_0^1} f = \frac{1}{3}$$

$\therefore f$ is integrable on $[0, 1]$ and $\int_0^1 f = \frac{1}{3}$

Remark: (i) $\forall P \in \mathcal{P}[a, b]$, $L(P, f) \le U(P, f)$, equality occurs when f is constant in $[a, b]$.

(ii) $w(P, f) = U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$, is called oscillatory sum of f corresponding to the partition P of $[a, b]$.

Theorem 2.2.1 **:** Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P is a partition of $[a, b]$. If Q be a refinement of P, then (i) $L(P, f) \le L(Q, f)$ (ii) $U(P, f) \ge U(Q, f)$

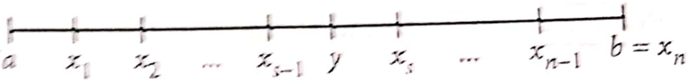
Proof: Let $P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$

Let P_1 be the refinement of P by adjoining one additional point y to P.

Let $P_1 = \{a = x_0 < x_1 < x_2 < \dots < x_{s-1} < y < x_s < \dots < x_n = b\}$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \dots, n$

Let $M_s' = \sup\limits_{x \in [x_{s-1}, y]} f(x)$



$M_s'' = \sup\limits_{x \in [y, x_s]} f(x)$, $m_s' = \inf\limits_{x \in [x_{s-1}, y]} f(x)$, $m_s'' = \inf\limits_{x \in [y, x_s]} f(x)$

Then $M_s', M_s'' \le M_s$ and $m_s', m_s'' \ge m_s$

$$\begin{aligned} \therefore L(P, f) - L(P_1, f) &= m_s(x_s - x_{s-1}) - m_s'(y - x_{s-1}) - m_s''(x_s - y) \\ &= m_s(x_s - y + y - x_{s-1}) - m_s'(y - x_{s-1}) - m_s''(x_s - y) \\ &= (m_s - m_s')(y - x_{s-1}) + (m_s - m_s'')(x_s - y) \\ &\le 0, \text{ since } m_s - m_s' \le 0, m_s - m_s'' \le 0 \text{ and } y - x_{s-1} > 0, x_s - y > 0 \end{aligned}$$

$\Rightarrow L(P, f) \le L(P_1, f)$ ... (1)

$$\begin{aligned} \text{Again, } U(P, f) - U(P_1, f) &= M_s(x_s - x_{s-1}) - M_s'(y - x_{s-1}) - M_s''(x_s - y) \\ &= (M_s - M_s')(y - x_{s-1}) + (M_s - M_s'')(x_s - y) \\ &\ge 0, \text{ since } M_s - M_s' \ge 0,\ M_s - M_s'' \ge 0 \text{ and} \\ &\qquad y - x_{s-1} > 0, x_s - y > 0 \end{aligned}$$

$\Rightarrow U(P, f) \ge U(P_1, f)$ ... (2)

Let P_2 be the refinement of P_1 by adjoining one additional pt. to P_1 then by above,

$L(P_1, f) \le L(P_2, f)$ ... (3) and $U(P_1, f) \ge U(P_2, f)$ ... (4)

$\therefore$ From (1) and (3), $L(P, f) \le L(P_2, f)$ and from (2) and (4), $U(P, f) \ge U(P_2, f)$

and on so. $\therefore$ if Q be a refinement of P, then $Q = P_n$ for some n and then

$L(P, f) \le L(Q, f)$ [(i) proved]

$U(P, f) \ge U(Q, f)$ [(ii) proved]

Remark: (i) If we make a refinement of a partition, then lower sums increases and upper sum decreases.

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(ii) Let P be a partition of $[a, b]$ and Q is a refinement of P. Then $L(P, f) \leq L(Q, f)$,
$U(P, f) \geq U(Q, f)$
$\Rightarrow w(Q, f) = U(Q, f) - L(Q, f) \leq U(P, f) - L(P, f) = w(P, f)$
$\therefore w(Q, f) \leq w(P, f)$

Theorem 2.2.2 : Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P and Q are two partitions of $[a, b]$. Then

(i) $L(P, f) \leq U(Q, f)$ (ii) $L(Q, f) \leq U(P, f)$

Proof : Let $P^* = P \cup Q$. Then P^* is a common refinement of P and Q.
$\therefore L(P, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(Q, f)$
$\Rightarrow L(P, f) \leq U(Q, f)$ [(i) proved]
Again, $L(Q, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f)$
$\therefore L(Q, f) \leq U(P, f)$ [(ii) proved]

Remark : For a bounded function f on $[a, b]$, no lower sum can be greater than any upper sum.

Theorem 2.2.3 : Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Proof : Let P, Q be any two partitions of $[a, b]$.
Making Q fixed, we get $L(P, f) \leq U(Q, f)$, $\forall P \in \mathcal{P}[a, b]$
$\Rightarrow \{L(P, f) : P \in P[a, b]\}$ is bounded above and $U(Q, f)$ is an upper bound of this set.
$\sup\{L(P, f) : P \in P[a, b]\} \leq U(Q, f)$

$\Rightarrow \underline{\int_a^b} f \leq U(Q, f)$. This holds for every $Q \in \mathcal{P}[a, b]$

$\Rightarrow \underline{\int_a^b} f$ is a lower bound of the set $\{U(Q, f) : Q \in \mathcal{P}[a, b]\}$

$\Rightarrow \underline{\int_a^b} f \leq \inf\{U(Q, f) : Q \in \mathcal{P}[a, b]\}$

$\Rightarrow \underline{\int_a^b} f \leq \overline{\int_a^b} f$ (Proved)

Theorem 2.2.4 : Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P is a partition of $[a, b]$. Let P_k be a refinement of P by adjoining k additional point to P and $||P|| \leq \delta$, $\delta > 0$.

Then (i) $0 \leq U(P, f) - U(P_k, f) \leq (M - m)\, k\, \delta$
(ii) $0 \leq L(P_k, f) - L(P, f) \leq (M - m)\, k\, \delta$

where M, m are respectively the suprimum and infimum of f on $[a, b]$.

Proof : Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$
Let P_1 be the refinement of P by adjoining one additional point y to P.
Let $P_1 = \{a = x_0 < x_1 < x_2 < \ldots < x_{s-1} < y < x_s < \ldots < x_n = b\}$
Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$.

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Let $M_s' = \sup_{x \in [x_{s-1}, y]} f(x)$, $m_s' = \inf_{x \in [x_{s-1}, y]} f(x)$

$M_s'' = \sup_{x \in [y, x_s]} f(x)$, $m_s'' = \inf_{x \in [y, x_s]} f(x)$

Then $m \leq m_s \leq m_s' \leq M_s' \leq M_s \leq M$ ... (1)

and $m \leq m_s \leq m_s'' \leq M_s'' \leq M_s \leq M$ ... (2)

(i) Now $U(P, f) - U(P_1, f) = M_s(x_s - x_{s-1}) - M_s'(y - x_{s-1}) - M_s''(x_s - y)$

$= (M_s - M_s')(y - x_{s-1}) + (M_s - M_s'')(x_s - y)$ ... (3)

Now from (1) and (2),

$0 \leq M_s - M_s' \leq M - m$ and $0 \leq M_s - M_s'' \leq M - m$

Also $y - x_{s-1} > 0$, $x_s - y > 0$

$\therefore$ from (3), $0 \leq U(P, f) - U(P_1, f) \leq (M - m)(y - x_{s-1}) + (M - m)(x_s - y)$

$= (M - m)(x_s - x_{s-1})$

$\leq (M - m)\,||P||$

$\leq (M - m)\,\delta$

Now if P_2 be the refinement of P_1 by adjoining one additional point to P_1, P_3 is a refinement of P_2 by adjoining one additional point to P_2 and so on, then by similar as above,

$0 \leq U(P_1, f) - U(P_2, f) \leq (M - m)\,\delta$

$0 \leq U(P_2, f) - U(P_3, f) \leq (M - m)\,\delta$

... ...

... ...

$U(P_{k-1}, f) - U(P_k, f) \leq (M - m)\,\delta$

Adding last k relations, we get

$0 \leq U(P, f) - U(P_k, f) \leq (M - m)\,k\,\delta$ (Proved)

(ii) $L(P_1, f) - L(P, f) = m_s'(y - x_{s-1}) + m_s''(x_s - y) - m_s(x_s - x_{s-1})$

$= (m_s' - m_s)(y - x_{s-1}) + (m_s'' - m_s)(x_s - y)$ ... (4)

From (1) and (2), $0 \leq m_s' - m_s \leq M - m$, $0 \leq m_s'' - m_s \leq M - m$

Also $y - x_{s-1} > 0$ and $x_s - y > 0$

$\therefore$ from (4), $0 \leq L(P_1, f) - L(P, f) \leq (M - m)(y - x_{s-1}) + (M - m)(x_s - y)$

$= (M - m)(x_s - x_{s-1})$

$\leq (M - m)\,||P||$

$\leq (M - m)\,\delta$

Now Let P_i be the refinement of P_{i-1} $(i = 2, 3, \ldots, k)$ by adjoining one additional point to P_{i-1}. Then on similar manner as above, we get

$0 \leq L(P_i, f) - L(P_{i-1}, f) < (M - m)\,\delta$, $i = 2, 3, \ldots, k$

Adding k relations, we get

$0 \leq L(P_k, f) - L(P, f) \leq (M - m)\,k\,\delta$ (Proved)

Remark : Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$.
Then $\exists\, B > 0$ such that $|f(x)| \le B \ \forall\, x \in [a, b]$

$\Rightarrow -B \le f(x) \le B \ \forall\, x \in [a, b]$

$\Rightarrow -B \le m \le f(x) \le M \le B \ \forall\, x \in [a, b]$

$\Rightarrow M - m \le 2B$

$\therefore$ the previous theorem becomes.

$0 \le U(P, f) - U(P_k, f) \le 2Bk\delta$ and $0 \le L(P_k, f) - L(P, f) \le 2Bk\delta$.

***Theorem 2.2.5* : (Darboux Theorem)**

Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then for given any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

(i) $U(P, f) < \overline{\int_a^b} f + \varepsilon \ \forall\, P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

(ii) $L(P, f) > \underline{\int_a^b} f - \varepsilon \ \forall\, P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

Proof : (i) $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$

$\Rightarrow \exists\, B > 0$ such that $|f(x)| \le B \ \forall\, x \in [a, b]$

Let $\varepsilon > 0$ be arbitrary.

Now $\overline{\int_a^b} f = \inf\{U(P, f) : P \in \mathcal{P}[a, b]\}$

$\Rightarrow \exists\, P_1 \in \mathcal{P}[a, b]$ such that $U(P_1, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$ ... (1)

Let $P_1 = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$

Let $\delta = \dfrac{\varepsilon}{4B(n-1)}$ and $P \in \mathcal{P}[a, b]$ such that $\|P\| < \delta$

Let $P^* = P \cup P_1$. Then P^* is a common refinement of P and P_1. P^* is a refinement o
by adjoining the $(n - 1)$ points $x_1, x_2, \ldots, x_{n-1}$ atmost.

$\therefore 0 \le U(P, f) - U(P^*, f) < 2B(n - 1)\delta$ [by theorem 2.2.4]

$\Rightarrow U(P, f) < U(P^*, f) + 2B(n - 1)\delta$

$= U(P^*, f) + \frac{\varepsilon}{2}$ ... (2)

Again P^* is a refinement of $P_1 \Rightarrow U(P^*, f) \le U(P_1, f)$

$\therefore$ from (2), $U(P, f) < U(P_1, f) + \frac{\varepsilon}{2}$

$< \overline{\int_a^b} f + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ [by (1)]

$= \overline{\int_a^b} f + \varepsilon$

$\therefore U(P, f) < \overline{\int_a^b} f + \varepsilon, \ \forall\, P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

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(ii) $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$

$\Rightarrow \exists\, B > 0$ s.t. $|f(x)| \le B \;\forall\, x \in [a, b]$.

Let $\varepsilon > 0$ be arbitrary.

Now $\underline{\int_a^b} f = \sup\{L(P, f) : P \in \mathcal{P}[a, b]\}$

$\Rightarrow \exists\, P_1 \in \mathcal{P}[a, b]$ such that $L(P_1, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}$ ... (1)

Let $P_1 = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$.

Let $\delta = \dfrac{\varepsilon}{4B(n-1)}$ and $P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

Let $P^* = P \cup P_1$. Then P^* is a common refinement of P and P_1.

Now the refinement P^* of P is obtained from P by adjoining the $(n-1)$ points $x_1, x_2, \ldots, x_{n-1}$ atmost.

$\therefore 0 \le L(P^*, f) - L(P, f) < 2B(n-1)\delta$ [by theorem 2.4]

$\Rightarrow L(P, f) > L(P^*, f) - 2B(n-1)\delta$

$= L(P^*, f) - \frac{\varepsilon}{2}$ ... (2)

Now P^* is a refinement of $P_1 \Rightarrow L(P^*, f) \ge L(P_1, f)$

$\therefore$ from (2), $L(P, f) > L(P_1, f) - \frac{\varepsilon}{2}$

$> \underline{\int_a^b} f - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}$, by (1)

$= \underline{\int_a^b} f - \varepsilon$

$\therefore L(P, f) > \underline{\int_a^b} f - \varepsilon \;\forall\, P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$.

***Definition 2.2.7*:** Let ϕ be a real valued function defined on $\mathcal{P}[a, b]$. Then $\phi(P)$ tends to a limil A as $\|P\| \to 0$ if for given any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|\phi(P) - A| < \varepsilon \;\forall\, P \in \mathcal{P}[a, b] \text{ satisfying } \|P\| < \delta$$

We write $\lim\limits_{\|P\| \to 0} \phi(P) = A$

Remark: (Another form of Darboux theorem)

Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then

(i) $\lim\limits_{\|P\| \to 0} U(P, f) = \overline{\int_a^b} f$ (ii) $\lim\limits_{\|P\| \to 0} L(P, f) = \underline{\int_a^b} f$

Proof: (i) First we prove the previous theorem (Th. 2.2.5) and obtained $U(P, f) < \overline{\int_a^b} f + \varepsilon \;\forall\, P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

$$\Rightarrow \left| U(P, f) - \overline{\int_a^b} f \right| < \varepsilon \;\forall\, P \in \mathcal{P}[a, b] \text{ satisfying } \|P\| < \delta$$

By the Condition of integrability

Case II: Let f is monotone decreasing.

We can prove similarly that f is integrable on $[a,b]$ in this case also.

This completes the proof.

Examples

① Let $f : [-1,1] \to \mathbb{R}$ be defined by

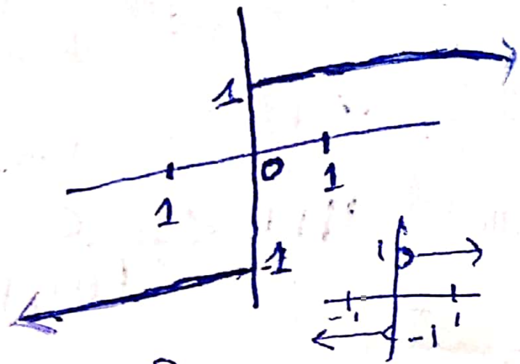
$f(x) = \operatorname{sgn} x$, $x \in [-1,1]$

Prove that f is integrable on $[-1,1]$

$\operatorname{sgn} x$
$= 1$, when $x >$
$= 0$, when $x =$
$= -1$, when x

$\frac{|x|}{x}$

$\Rightarrow f(x) = 1$ for $0 < x \leq 1$.
$= 0$ for $x = 0$.
$= -1$ for $-1 \leq x < 0$.



f is monotone increasing on $[-1,1]$.

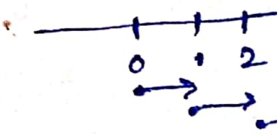
Therefore f is integrable on $[-1,1]$.

② Let $f : [0,3] \to \mathbb{R}$ be defined by

$f(x) = [x]$, $x \in [0,3]$.

Prove that f is integrable on $[0,3]$.

$\Rightarrow f(x) = 0$ when $0 \leq x < 1$
$= 1$ when $1 \leq x < 2$
$= 2$ when $2 \leq x < 3$



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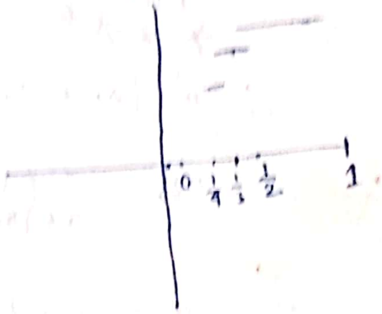
f is monotone increasing on $[0,1]$.
Therefore f is integrable on $[0,1]$.

CH'97
(b) Let $f:[0,1]\to\mathbb{R}$ be defined by
$f(x) = \frac{1}{n}$, $\frac{1}{n+1} < x \le \frac{1}{n}$, $n=1,2,3,\ldots$

and $f(0)=0$.

Prove that f is integrable on $[0,1]$.

$\Rightarrow$ $f(x) = 0$, $x = 0$
$= 1$, $\frac{1}{2} < x \le 1$
$= \frac{1}{2}$, $\frac{1}{3} < x \le \frac{1}{2}$
$= \frac{1}{3}$, $\frac{1}{4} < x \le \frac{1}{3}$



f is monotone increasing on $[0,1]$. Therefore f is integrable on $[0,1]$.

Thm CH. 2001 CH'93

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$.
Then f is integrable on $[a,b]$. $\|P\| < \delta$ V.V.I

proof: Since f is continuous on $[a,b]$, f is bounded on $[a,b]$.

Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$.

is not true.

For example, let $f(x) = [x]$, $x \in [0,3]$.

Then f is not continuous on $[0,3]$ but f is integrable on $[0,3]$

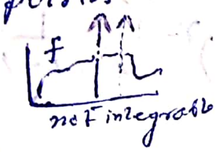
f is discontinuous at 1,2,3. | যেহেতু 1,2,3 তিনটা point-এ function টা ভাঙা!!

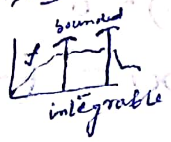
**

The class of function continuous on $[a,b]$ is denoted by $C[a,b]$. $C[a,b]$ is a proper subset of $R[a,b]$.

Th^m (CH'90)

Let $f:[a,b] \to R$ be bounded on $[a,b]$ and continuous on $[a,b]$ except at a finite number of points on $[a,b]$. Then f is integrable on $[a,b]$.





Th^m

Let $f:[a,b] \to R$ be bounded on $[a,b]$ and continuous on $[a,b]$ except at a set of points S such that the number of limit points of S is finite. Then f is integrable on $[a,b]$.

C.H. C.H. C.H

Let $f:[0,3] \to R$ be defined by

$f(x) = [x]$, $x \in [0,3]$

Prove that f is integrable on $[0,3]$

$\Rightarrow$ $f(x) = 0$, when $0 \le x < 1$.

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$= 1$ when $1 \le x < 2$

$= 2$ when $2 \le x < 3$

$= 3$ " $x = 3$.

f is bounded on $[0,3]$.

f is continuous on $[0,3]$ except at the points $1, 2, 3$.

Since the number of points of discontinuity is finite, f is integrable on $[0,3]$.

V.V. 2001 ② C.H. C.H. C.H. C.H '94

A function f is defined on $[0,1]$ by

$$f(0) = 0$$
$$f(x) = (-1)^{r-1}, \quad \frac{1}{r+1} < x \le \frac{1}{r}$$
$$r = 1, 2, 3 \ldots n.$$

$\Rightarrow f(0) = 0$

$f(x) = 1$, when $\frac{1}{2} < x \le 1$

$= -1$, " $\frac{1}{3} < x \le \frac{1}{2}$

$= 1$ " $\frac{1}{4} < x \le \frac{1}{3}$

$= - -$

$= - -$

f is bounded on $[0,1]$. f is continuous on $[0,1]$, except at the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

The set of points of discontinuity