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E-learning Materials

SEM-4, Unit-1

CC-8

Topic - Riemann Integration.

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But $\overline{\int_0^1} f = \overline{\int_0^1} g = 1$ and $\underline{\int_0^1} f = \underline{\int_0^1} g = -1$

$\Rightarrow$ neither f nor g is integrable on $[0, 1]$

Theorem 2.4.3 **:** Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then $|f| \in \mathcal{R}[a, b]$

Proof : $f \in \mathcal{R}[a, b]$

$\Rightarrow f$ is bounded on $[a, b]$

$\Rightarrow \exists\, B > 0$ such that $|f(x)| < B\ \forall\, x \in [a, b]$

$\therefore\ \big||f|(x)\big| = |f(x)| < B,\ \forall\, x \in [a, b]$

$\Rightarrow |f|$ is bounded on $[a, b]$

Let $\varepsilon > 0$ be arbitrary

Since $f \in \mathcal{R}[a, b]$, $\exists\, P = \{a = x_0 < x_1 < \ldots < x_n = b\} \in \mathcal{P}[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon \qquad \ldots (1)$$

Let $\quad M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\qquad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$\quad M_r' = \sup\limits_{x \in [x_{r-1}, x_r]} |f|(x)$, $\qquad m_r' = \inf\limits_{x \in [x_{r-1}, x_r]} |f|(x)$, $r = 1, 2, \ldots, n$

Now, $\forall\, x, y \in [x_{r-1}, x_r]$ $(r = 1, 2, \ldots, n)$

$$\big||f|(x) - |f|(y)\big| = \big||f(x)| - |f(y)|\big|$$

$$\leq |f(x) - f(y)| \quad [\because\ ||a| - |b|| \leq |a - b|]$$

$\Rightarrow \sup\{\big||f|(x) - |f|(y)\big| : x, y \in [x_{r-1}, x_r]\}$

$\leq \sup\{|f(x) - f(y)| : x, y \in [x_{r-1}, x_r]\}$

$\Rightarrow M_r' - m_r' \leq M_r - m_r$, $r = 1, 2, \ldots, n$

$$\Rightarrow \sum_{r=1}^{n} (M_r' - m_r')(x_r - x_{r-1}) \leq \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$

$\Rightarrow U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon$

$\Rightarrow |f| \in \mathcal{R}[a, b]$, by a sufficient condition of integrability.

Remark : $|f| \in \mathcal{R}[a, b]$ may not imply that $f \in \mathcal{R}[a, b]$

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = 1, \quad x \in [0, 1] \cap \mathbb{Q}$$
$$= -1, \quad x \in [0, 1] - \mathbb{Q}$$

Then $|f(x)| = 1$, $\forall\, x \in [0, 1]$

$\Rightarrow |f|$ is constant in $[0, 1]$

$\Rightarrow |f| \in \mathcal{R}[0, 1]$

But $f \notin \mathcal{R}[0, 1]$, as $\overline{\int_0^1} f = 1$ and $\underline{\int_0^1} f = -1$

$$\therefore U(P, gof) - L(P, gof) = \sum_{r=1}^{n} (M_r' - m_r')(x_r - x_{r-1})$$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, by (iii) and (iv)

$\Rightarrow gof \in \mathcal{R}[a, b]$

Remark : If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $g : [c, d] \to \mathbb{R}$ be integrable on $[c, d]$, where $f([a, b]) \subset [c, d]$, then gof may not be integrable on $[a, b]$.

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$f(x) = 0$, if $x = 0$ or x is irrational

$= \frac{1}{n}$, if $x = \frac{m}{n}$, where $m, n \in \mathbb{N}$ and $gcd(m, n) = 1$

Then $f \in \mathcal{R}[0, 1]$

Let $g : [0, 1] \to \mathbb{R}$ be defined by

$g(x) = 1$, if $x \neq 0$

$= 0$, if $x = 0$

Then g is bounded on $[0, 1]$ and continuous on $[0, 1]$ except at $x = 0$

$\therefore g$ is integrable on $[0, 1]$

Now $gof(x) = 0$, if $x = 0$ or x irrational

$= 1$, if x rational

Clearly gof is not integrable on $[0, 1]$

Theorem 2.4.12 : Let $f : [a, b] \to \mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$ be both bounded on $[a, b]$ such that $f(x) = \phi(x)$ except for a finite number of points in $[a, b]$. If f be integrable on $[a, b]$ then ϕ is integrable on $[a, b]$ and $\int_a^b \phi = \int_a^b f$.

Proof : ϕ is bounded on $[a, b]$

$\Rightarrow \exists\, k > 0$ such that $|\phi(x)| < k$, $\forall\, x \in [a, b]$

Let $f(x) = \phi(x)$ on $[a, b]$, except for m points $x_1, x_2, \ldots, x_m$

where $a \le x_1 < x_2 < \ldots < x_m \le b$

Case 1. Let $a < x_1 < x_2 < \ldots < x_m < b$

Let $\varepsilon > 0$ arbitrary.

Let us enclose $x_1, x_2, \ldots, x_m$ by m non-overlapping sub-intervals $\left[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}\right]$, $\left[x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}\right], \ldots, \left[x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}\right]$

where $a < x_1 - \frac{\delta_1}{2}$, $x_m + \frac{\delta_m}{2} < b$

and $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\varepsilon}{4k}$ ... (i)

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Let $M_r = \sup_{x\in\left[x_r-\frac{\delta_r}{2},\, x_r+\frac{\delta_r}{2}\right]} \phi(x)$, $m_r = \inf_{x\in\left[x_r-\frac{\delta_r}{2},\, x_r+\frac{\delta_r}{2}\right]} \phi(x)$, $r = 1, 2, \ldots, m$

Then $M_r - m_r \le 2k$, $r = 1, 2, \ldots, m$ ... (ii)

Now, $f = \phi$ on the remaining $(m + 1)$ subintervals $\left[a, x_1-\frac{\delta_1}{2}\right], \left[x_1+\frac{\delta_1}{2}, x_2-\frac{\delta_2}{2}\right], \ldots, \left[x_m+\frac{\delta_m}{2}, b\right]$

$\Rightarrow$ ϕ is integrable on these $m + 1$ subintervals.

$\Rightarrow$ $\exists$ a partition P_1 of $\left[a, x_1-\frac{\delta_1}{2}\right]$, P_2 of $\left[x_1+\frac{\delta_1}{2}, x_2-\frac{\delta_2}{2}\right]$, ..., P_{m+1} of $\left[x_m+\frac{\delta_m}{2}, b\right]$

such that ... (iii)

$U(P_i, \phi) - L(P_i, \phi) < \frac{\varepsilon}{2(m+1)}$, $i = 1, 2, \ldots, (m + 1)$

Let $P = P_1 \cup P_2 \cup \ldots \cup P_{m+1}$

Then P is a partition of $[a, b]$

$$U(P, \phi) - L(P, \phi) = \sum_{i=1}^{m+1}[U(P_i, \phi)-L(P_i, \phi)] + \sum_{r=1}^{m}(M_r - m_r)\,\delta_r$$

$$< \sum_{i=1}^{m+1}\frac{\varepsilon}{2(m+1)} + 2k\sum_{r=1}^{m}\delta_r \text{, by (ii) and (iii)}$$

$$< \frac{\varepsilon}{2(m+1)}\cdot(m+1) + 2k\cdot\frac{\varepsilon}{4k} \text{, by (i)}$$

$$= \varepsilon$$

$\Rightarrow$ ϕ is integrable on $[a, b]$, by a sufficient condition.

Case 2. Either $a = x_1$ or $b = x_m$ or both

If $a = x_1$, the subinterval x_1 can be taken as $[a, a + \delta_1]$ and if $b = x_m$, the subinterval x_m can be taken as $[b - \delta_m, b]$.

In any case, proceeding with similar arguments it can be proved that ϕ is integrable on $[a, b]$.

2nd part : Let $g(x) = f(x) - \phi(x)$, $x \in [a, b]$

Then g is integrable on $[a, b]$, as f and ϕ are so.

Also $g(x) = 0$ on $[a, b]$ except for m points $x_1, x_2, \ldots, x_m$

Consider $g_+(x) = \frac{1}{2}\{g(x) + |g(x)|\}$, $x \in [a, b]$

and $g_-(x) = \frac{1}{2}\{g(x) - |g(x)|\}$, $x \in [a, b]$

Then $g(x) = g_+(x) + g_-(x)$, $x \in [a, b]$ ... (iv)

Since g and $|g|$ are integrable on $[a, b]$, both g_+ and g_- are integrable on $[a, b]$.

Let $P = \{a = y_0 < y_1 < y_2 < \ldots < y_n = b\} \in \mathcal{P}[a, b]$

Let $m'_r = \inf_{x\in[y_{r-1}, y_r]} g_+(x)$, $r = 1, 2, \ldots, n$

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Since $g_+(x) = 0$ on $[a, b]$, except for at most m points, $m'_r = 0, r = 1, 2, ..., n.$

$$\therefore L(P, g_+) = \sum_{r=1}^{n} m'_r (y_r - y_{r-1}) = 0$$

$$\therefore \underline{\int_a^b} g_+(x)\, dx = \sup_{P \in \mathbb{P}[a,b]} L(P, g_+) = 0$$

$\Rightarrow \int_a^b g_+(x)\, dx = 0$, since g_+ is integrable on $[a, b]$

Similarly, it can be shown that $\int_a^b g_-(x)\, dx = 0$

$\therefore$ from (iv), $\int_a^b g(x)\, dx = \int_a^b g_+(x)\, dx + \int_a^b g_-(x)\, dx = 0$

$$\Rightarrow \int_a^b \{f(x) - \phi(x)\}\, dx = 0$$

$\Rightarrow \int_a^b f(x)\, dx = \int_a^b \phi(x)\, dx$ (Proved)

Remark: If f and ϕ be both bounded on $[a, b]$ and $f(x) = \phi(x)$ on $[a, b]$, except for an enumerable number of points of $[a, b]$ and if f is integrable on $[a, b]$, then ϕ may not be integrable on $[a, b]$.

Let $f(x) = 1, x \in [0, 1]$ and $\phi(x) = \begin{cases} 0, & x \in [0,1] \cap \mathbb{Q} \\ 1, & x \in [0,1] - \mathbb{Q} \end{cases}$

Then f and ϕ are both bounded on $[0, 1]$ and $f = \phi$ on $[0, 1]$ except for enumerable subset of $[0, 1]$ and f is integrable on $[0, 1]$, but ϕ is not integrable on $[0, 1]$.

SOLVED PROBLEMS

Problem 13. Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$.

Prove that (i) max $(f, g) : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

(ii) min $(f, g) : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

Solution: (i) max $(f, g)(x) = \max \{f(x), g(x)\}, x \in [a, b]$

$\therefore$ max $(f, g) = \frac{1}{2} [(f + g) + |f - g|]$

Now $f, g \in \mathcal{R}[a, b] \Rightarrow f \pm g \in \mathbb{R}[a, b]$

$\Rightarrow f + g \in \mathcal{R}[a, b]$ and $|f - g| \in \mathcal{R}[a, b]$

$\Rightarrow \frac{1}{2} [(f + g) + |f - g|] \in \mathcal{R}[a, b]$

$\therefore$ max $(f, g) \in \mathcal{R}[a, b]$

(ii) ***Hint.*** min $(f, g) = \frac{1}{2} [(f + g) + |f - g|]$

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Problem 14. Let $f: [a, b] \to \mathbb{R}$ be integrable on $[a, b]$.

Prove that $f^+ : [a, b] \to \mathbb{R}$, defined by $f^+(x) = f(x)$, if $f(x) \geq 0$, $x \in [a, b]$

$= 0$, if $f(x) < 0$, $x \in [a, b]$

is integrable on $[a, b]$.

Solution : $f^+ : [a, b] \to \mathbb{R}$ is defined by $f^+(x) = f(x)$, if $f(x) \geq 0$, $x \in [a, b]$

$= 0$, if $f(x) < 0$, $x \in [a, b]$

Then $f^+(x) = \max\{f(x), 0\}$, $x \in [a, b]$

$\Rightarrow f^+ = \frac{1}{2}\{f + |f|\}$

Now $f \in \mathcal{R}[a, b] \Rightarrow |f| \in \mathcal{R}[a, b]$

$\therefore \frac{1}{2}\{f + |f|\} \in \mathcal{R}[a, b]$

$\Rightarrow f^+ \in \mathcal{R}[a, b]$

Alternative proof :

Let $\varepsilon > 0$ be arbitrary

Since $f \in \mathcal{R}[a, b]$, $\exists\, P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\} \in \mathcal{P}[a, b]$

Such that $U(P, f) - L(P, f) < \varepsilon$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} f^+(x)$ $\quad m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} f^+(x)$, $r = 1, 2, \ldots, n$

Now, $f^+(x) = f(x)$, if $f(x) \geq 0$, $x \in [a, b]$

$= 0$, if $f(x) < 0$, $x \in [a, b]$

$\therefore$ if $0 \leq m_r \leq M_r$, then $M'_r = M_r$, $m'_r = m_r$ and so $M'_r - m'_r = M_r - m_r$

and if $m_r \leq 0 \leq M_r$, then $M'_r = M_r$, $m'_r \geq m_r$ and so $M'_r - m'_r \leq M_r - m_r$

and if $m_r \leq M_r \leq 0$, then $M'_r = m'_r = 0$ and so $M'_r - m'_r \leq M_r - m_r$

$$\therefore U(P, f^+) - L(P, f^+) = \sum_{r=1}^{n} (M'_r - m'_r)(x_r - x_{r-1})$$

$$\leq \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$

$$= U(P, f) - (P, f)$$

$$< \varepsilon$$

$\Rightarrow f^+$ is integrable on $[a, b]$.

(6)

Problem 15. Let $f: [-100, 100] \to \mathbb{R}$ be defined by

$$f(x) = 0,\ x \in [-100, 100] \cap \mathbb{Z}$$
$$= 1,\ x \in [-100, 100] - \mathbb{Z}$$

Evaluate $\int_{-100}^{100} f$, if exist.

Solution : $f(x) = 0,\ x \in [-100, 100] \cap \mathbb{Z}$

$= 1,\ x \in [-100, 100] - \mathbb{Z}$

f is bounded on $[-100, 100]$

Let $g(x) = 1,\ x \in [-100, 100]$. Then g is integrable on $[-100, 100]$

Now $g = f$ on $[-100, 100]$ except for finite number of points, namely at the points $0, \pm 1, \pm 2, \ldots, \pm 100$.

$\therefore f$ is integrable on $[-100, 100]$

and $\int_{-100}^{100} f = \int_{-100}^{100} g = 1(100 + 100) = 200$

Problem 16. Let $f: [0, 3] \to \mathbb{R}$ be defined by $f(x) = x - [x],\ x \in [0, 3]$. Show that $f \in \mathcal{R}[0, 3]$.

Evaluate $\int_0^3 f$.

Solution : $f(x) = x - [x],\ x \in [0, 3]$

Then $f(x) = x,\ 0 \le x < 1$

$= x - 1,\ 1 \le x < 2$

$= x - 2,\ 2 \le x < 3$

$= 0,\ x = 3$

Since $|f(x)| < 1\ \forall\ x \in [0, 3]$, f is bounded on $[0, 3]$. f is continuous on $[0, 3]$ except at finite number of points, namely at the points 1, 2 and 3.

$\therefore f$ is integrable on $[0, 3]$.

Let $\phi_1(x) = x,\ x \in [0, 1]$

$\phi_2(x) = x - 1,\ x \in [1, 2]$

and $\phi_3(x) = x - 2,\ x \in [2, 3]$

Then $\phi_1 = f$ on $[0, 1]$ except at $x = 1$, $\therefore \int_0^1 f = \int_0^1 \phi_1$

$\phi_2 = f$ on $[1, 2]$ except at $x = 2$, $\therefore \int_1^2 f = \int_1^2 \phi_2$

and $\phi_3 = f$ on $[2, 3]$ except at $x = 3$, $\therefore \int_2^3 f = \int_2^3 \phi_3$

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$$\therefore \int_0^3 f = \int_0^1 f + \int_1^2 f + \int_2^3 f = \int_0^1 \phi_1 + \int_1^2 \phi_2 + \int_2^3 \phi_3$$

$$= \int_0^1 x\,dx + \int_1^2 (x-1)\,dx + \int_2^3 (x-2)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$$

$$= \frac{1}{2} + \left[-\frac{1}{2} + 1\right] + \left[\frac{9}{2} - 6 - 2 + 4\right]$$

$$= \frac{3}{2}$$

Problem 17. A function $f : [0, 1] \to \mathbb{R}$ is defined by $f(0) = 1$, $f(x) = (-1)^{n-1}$, if $\frac{1}{n+1} < x \le \frac{1}{n}$ $(n = 1, 2, \ldots)$. Prove that $f \in \mathcal{R}[0, 1]$. Evaluate $\int_0^1 f$.

Solution : $f(0) = 1$,

$$f(x) = (-1)^{n-1}, \text{ if } \frac{1}{n+1} < x \le \frac{1}{n},\ n = 1, 2, \ldots$$

Since $|f(x)| = 1, \forall x \in [0, 1]$, f is bounded on $[0, 1]$

Now $f(x) = 1, \ \frac{1}{2} < x \le 1$

$= -1, \ \frac{1}{3} < x \le \frac{1}{2}$

$= 1, \ \frac{1}{4} < x \le \frac{1}{3}$

... ...

... ...

$= 1, x = 0$

f is continuous on $[0, 1]$ except for infinite number of points $0, \frac{1}{2}, \frac{1}{3}, \ldots$, but the set of all discontinuities have only one limit point, namely 0.

$\therefore f$ is integrable on $[0, 1]$.

Let $\phi_i(x) = (-1)^{i-1}, \ \frac{1}{i+1} \le x \le \frac{1}{i}, i = 1, 2, \ldots$

Then $\phi_i = f$ on $\left[\frac{1}{i+1}, \frac{1}{i}\right]$ except at one point $\frac{1}{i+1}$ $(i = 1, 2, \ldots)$

$$\int_{\frac{1}{i+1}}^{\frac{1}{i}} f = \int_{\frac{1}{i+1}}^{\frac{1}{i}} \phi_i = (-1)^{i-1}\left(\frac{1}{i} - \frac{1}{i+1}\right), i = 1, 2, \ldots$$

$$\int_0^1 f = \int_{\frac{1}{2}}^1 f + \int_{\frac{1}{3}}^{\frac{1}{2}} f + \int_{\frac{1}{4}}^{\frac{1}{3}} f + \int_{\frac{1}{5}}^{\frac{1}{4}} f + \cdots$$

$$= \int_{\frac{1}{2}}^1 \phi_1 + \int_{\frac{1}{3}}^{\frac{1}{2}} \phi_2 + \int_{\frac{1}{4}}^{\frac{1}{3}} \phi_3 + \int_{\frac{1}{5}}^{\frac{1}{4}} \phi_4 + \cdots$$

(8)

$= (-1)^0\left(1-\frac{1}{2}\right)+(-1)^1\left(\frac{1}{2}-\frac{1}{3}\right)+(-1)^2\left(\frac{1}{3}-\frac{1}{4}\right)+(-1)^3\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$

$= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

$= 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)-1$

$= 2\log 2 - \log e\ \left[\log(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots, -1 < x \le 1\right]$

$= \log\left(\frac{4}{e}\right)$

Problem 18. Let $f: [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0$, $f(x) = a^n$, $a^{n+1} < x \le a^n$, $n = 0, 1, 2, \ldots$ $(0 < a < 1)$. Show that f is integrable on $[0, 1]$ and $\int_0^1 f = \frac{1}{1+a}$.

Solution : $f: [0, 1] \to \mathbb{R}$ is defined by $f(0) = 0$, $f(x) = a^n$, $a^{n+1} < x \le a^n$, $n = 0, 1, 2, \ldots$ $(0 < a < 1)$.

Then $f(x) = 1,\ a < x \le 1$

$= a,\ a^2 < x \le a$

$= a^2,\ a^3 < x \le a^2$

... ...

... ...

$= 0,\ x = 0$

f is bounded on $[0, 1]$ and is continuous on $[0, 1]$ except for the points $a, a^2, a^3, \ldots$, the set of points of discontinuities of f on $[0, 1]$ is infinite and the set have only one limit point, namely 0.

$\therefore$ f is integrable on $[0, 1]$

For $n = 0, 1, 2, \ldots$ let us define $g_n : [a^{n+1}, a^n] \to \mathbb{R}$ by $g_n(x) = a^n$, $a^{n+1} \le x \le a^n$.

Then for each $n = 0, 1, 2, \ldots$, $g_n = f$ on $[a^{n+1}, a^n]$, except at $x = a^{n+1}$

$$\therefore \int_{a^{n+1}}^{a^n} g_n(x)\,dx = \int_{a^{n+1}}^{a^n} f(x)\,dx$$

$$\therefore \int_0^1 f = \int_a^1 f + \int_{a^2}^a f + \int_{a^3}^a f + \cdots$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} \int_{a^{i+1}}^{a^i} f$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} \int_{a^{i+1}}^{a^i} g_i(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} a^i(a^i - a^{i+1})$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} (1-a)(a^2)^i$$

i.e. $\int_0^1 \frac{dx}{1+x+x^2} < \frac{\pi}{4}$ ——— (1)

Again In [0,1], $1+x+x^2 \le 3$

$\therefore \frac{1}{1+x+x^2} \ge \frac{1}{3}$.

$\therefore \int_0^1 \frac{dx}{1+x+x^2} \ge \int_0^1 \frac{dx}{3}$.

Let $\psi(x) = \frac{1}{3}$

At $x=0$ $f(x) = \frac{1}{1}$ and $\psi(x) = \frac{1}{3}$

$\therefore$ f & ψ are both continuous at 0 and $f(0) > \psi(0)$.

$\therefore \int_0^1 f(x)\,dx > \int_0^1 \psi(x)\,dx$

$= \int_0^1 \frac{1}{3}\,dx$

$= \frac{1}{3}$

$\therefore \int_0^1 \frac{dx}{1+x+x^2} > \frac{1}{3}$ ——— (2)

from (1) & (2) it follows that

$\frac{1}{3} < \int_0^1 \frac{dx}{1+x+x^2} < \frac{\pi}{4}$.

$\frac{10}{20}$ C.H. C.H. C.H. 2001.

Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$

$\Rightarrow$ In [0,1] $4-x^2+x^3 \ge 4-x^2$

or $\sqrt{4-x^2+x^3} \ge \sqrt{4-x^2}$

or $\frac{1}{\sqrt{4-x^2+x^3}} \leq \frac{1}{\sqrt{4-x^2}}$

$\therefore \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Let $f(x) = \frac{1}{\sqrt{4-x^2+x^3}}$ and $\phi(x) = \frac{1}{\sqrt{4-x^2}}$

At $x = \frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{\sqrt{4-\frac{1}{4}+\frac{1}{8}}}$, $\phi(\frac{1}{2}) = \frac{1}{\sqrt{4-\frac{1}{4}}}$

f and ϕ are both continuous at $\frac{1}{2}$ and $f(\frac{1}{2}) < \phi(\frac{1}{2})$.

$\therefore \int_0^1 f(x)\,dx < \int_0^1 \phi(x)\,dx$

or $\int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$ —— (1)

Again ~~Let $\Psi(x) =$~~

In $[0,1]$, $4-x^2+x^3 = 4-(x^2-x^3) \leq 4$.

$\therefore \sqrt{4-x^2+x^3} \leq 2$

or $\frac{1}{\sqrt{4-x^2+x^3}} \geq \frac{1}{2}$

$\therefore \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} \geq \int_0^1 \frac{1}{2}\,dx$

Let $\Psi(x) = \frac{1}{2}$

At $x = \frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{\sqrt{4-\frac{1}{4}+\frac{1}{8}}}$, $\Psi(\frac{1}{2}) = \frac{1}{2}$

$\therefore f(\frac{1}{2}) > \Psi(\frac{1}{2})$ and also f & g are both continuous at $\frac{1}{2}$

Particular case: Let $a = 0$, $b = 1$, Then $h = \frac{1}{n}$.

In this case

$$\int_0^1 f = \lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

(11)

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).$$

****** examination

Evaluate the limit $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$

$\Rightarrow$ Let $P = \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$.

Then $\log P = \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log\left(1+\frac{n}{n}\right)\right]$

$$= \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right).$$

$$\therefore \lim_{n\to\infty} \log P = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right), \quad \text{where } f(x) = \log(1+x)$$

$$= \int_0^1 f(x)\,dx,$$ Since f is integrable on $[0,1]$ as $\log(1+x)$ is continuous on $[0,1]$.

$$= \int_0^1 \log(1+x)\,dx$$

$$= \Big[\log(1+x)\cdot x\Big]_0^1 - \int_0^1 \frac{1}{1+x}\,x\,dx$$

$$= 1\cdot\log 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right)dx$$

$$= \log 2 - \Big[x - \log(1+x)\Big]_0^1$$

$= \log 2 - 1 + \log 2$

$= 2\log 2 - 1$

$= \log 4 - 1$

$= \log 4 - \log e$

$\therefore \lim_{n\to\infty} \log P = \log\left(\frac{4}{e}\right)$

$\log x$ function continuous হওয়ায়

$\lim_{n\to\infty} \log P = \log \lim_{n\to\infty} P$.

$\lim_{x\to c} f(x) = f(c)$ if f is continuous at c

$\lim_{x\to c} \log x = \log c$, Since $\log x$ is continuous at c.

or, $\lim_{x\to c} \log x = \log\left(\lim_{x\to c} x\right)$.

$\therefore \lim_{n\to\infty} P = \frac{4}{e}$, Since $\log x$ is Continuous. (12)

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(15) ① $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+3n}\right]$

Ans = 389

~~$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{3r}{n}}\right]$~~

~~$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1+\frac{3r}{n}}$~~

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