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E-Learning Materials

SEM-4, Unit-1

CC-8

Topic – Riemann Integration.

Date – 16.04.2020

Prepaired by – Dr. Alauddin Dafeda

$\Rightarrow \dfrac{1}{2-f}$ is integrable on [0, 1]

$\therefore g$ is integrable on [0, 1], where $g(x) = \dfrac{1}{2-f(x)}, x \in [0, 1]$

Thus the statement is true.

Problem 26. Give examples of a monotonic and a non-monotonic function on [0, 1] with infinitely many points of discontinuity such that the functions are bounded and Riemann integrable on [0, 1].

Solution : Let $f: [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0$,

$$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, n = 0, 1, 2, \ldots$$

Then f is bounded and monotonic increasing on [0, 1] and hence integrable on [0, 1]. f is discontinuous at the infinite number of points $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

Again, let $f: [0, 1] \to \mathbb{R}$ be defined by $f(0) = 1$,

$$f(x) = (-1)^{n-1}, \frac{1}{n+1} < x \leq \frac{1}{n} \ (n = 1, 2, \ldots)$$

Then f is bounded on [0, 1] and f is non-monotonic on [0, 1]. Since f is discontinuous at an infinite number of points $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, but the set of all discontinuities have only one limit point, namely '0', f is integrable on [0, 1].

Problem 27. A function $f: [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \frac{1}{2^n}, \text{ if } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, n = 0, 1, 2, \ldots$$
$$= 0 \quad , \quad x = 0$$

(i) Show that f is R-integrable on [0, 1]

(ii) Evaluate $\int_0^1 f$.

Solution : (i)
$$f(x) = 1, \quad \frac{1}{2} < x \leq 1$$
$$= \frac{1}{2}, \quad \frac{1}{2^2} < x \leq \frac{1}{2}$$
$$= \frac{1}{2^2}, \quad \frac{1}{2^3} < x \leq \frac{1}{2^2}$$
$$\ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots$$
$$= 0, x = 0$$

Clearly f is bounded on [0, 1], f is continuous on [0, 1] except for an infinite number of points in [0, 1], namely the points $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$, but the set of all discontinuities of f on [0, 1] has only one limit point, namely 0.

$\therefore f$ is integrable on [0, 1]

(2)

(ii) For $r = 0, 1, 2, \ldots$, we define $\phi_r : \left[\frac{1}{2^{r+1}}, \frac{1}{2^r}\right] \to \mathbb{R}$ by

$$\phi_r(x) = \frac{1}{2^r}, \, x \in \left[\frac{1}{2^{r+1}}, \frac{1}{2^r}\right].$$

Then $\phi_r = f$ on $\left[\frac{1}{2^{r+1}}, \frac{1}{2^r}\right]$ except at $\frac{1}{2^{r+1}}$ $(r = 0, 1, 2, \ldots)$

$$\therefore \int_{1/2^{r+1}}^{1/2^r} \phi_r(x)\,dx = \int_{1/2^{r+1}}^{1/2^r} f(x)\,dx, \; r = 0, 1, 2, \ldots$$

$$\therefore \int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{n} \int_{1/2^{r+1}}^{1/2^r} f(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \int_{1/2^{r+1}}^{1/2^r} \phi_r(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{2^r}\left(\frac{1}{2^r} - \frac{1}{2^{r+1}}\right)$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{2^r}\cdot\frac{1}{2^r}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{r=0}^{n} \left(\frac{1}{4}\right)^r$$

$$= \frac{1}{2} \lim_{n\to\infty} \left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n\right]$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}}$$

$$= \frac{2}{3}, \text{ since } \lim_{n\to\infty} \left(\frac{1}{4}\right)^{n+1} = 0$$

2.5 Some Inequalities on Riemann Integrals

Theorem 2.5.1 **:** Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and M, m are respectively the suprimum and infimum of f on $[a, b]$ then $m(b - a) \le \int_a^b f \le M(b - a)$.

Proof : Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\} \in \mathcal{P}[a, b]$

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$

Then $m \le m_r \le M_r \le M \quad \forall\, r = 1, 2, \ldots, n.$

$$\Rightarrow \sum_{r=1}^{n} m(x_r - x_{r-1}) \le \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M(x_r - x_{r-1})$$

$$\Rightarrow m(b - a) \le L(P, f) \le U(P, f) \le M(b - a) \quad \ldots \text{(i)}$$

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SOLVED PROBLEMS

Problem 28. Let f be non-negative continuous function on $[a, b]$ and $\int_a^b f = 0$. Then $f(x) = 0 \; \forall x \in [a, b]$.

Solution : f is non-negative and continuous on $[a, b]$ and $\int_a^b f = 0$.

If possible, let $\exists\, c \in [a, b]$ and $f(c) > 0$

Then f is non-negative, integrable on $[a, b]$ and at a point of continuity $c \in [a, b]$, $f(c) > 0$.

$\therefore \int_a^b f > 0$, a contradiction to the hypothesis that $\int_a^b f = 0$

$\therefore \nexists\, c \in [a, b]$ such that $f(c) > 0$

$\therefore f(x) = 0 \; \forall x \in [a, b]$

Problem 29. If a functin f is continuous on a closed interval $[a, b]$ and $\int fg = 0$ for every continuous function g on $[a, b]$, prove that $f(x) = 0$ for all $x \in [a, b]$

Solution : Let f be continuous on $[a, b]$ and $\int_a^b fg = 0$ for every continuous function g on $[a, b]$ ... (1)

Since (1) holds for every continuous function g on $[a, b]$, it also holds for $g = f$.

$\therefore$ (1) $\Rightarrow \int_a^b f^2 = 0$

Now $f^2(x) \geq 0 \; \forall x \in [a, b]$ and f^2 is continuous on $[a, b]$ and $\int_a^b f^2 = 0$

$\Rightarrow f^2(x) = 0, \forall x \in [a, b]$

$\Rightarrow f(x) = 0 \; \forall x \in [a, b]$. (Proved)

Problem 30. A function f is integrable on $[a, b]$ and $\int_a^b f^2(x)\,dx = 0$. Prove that $f(x) = 0$ at every point of continuity in $[a, b]$.

Solution : $f \in R[a, b]\, f^2 \in R[a, b]$.

Since f is integrable on $[a, b]$, f is continuous at some $c \in [a, b]$ [f is continuous a.e on $[a, b]$]

If possible, let $f(c) \neq 0$. Then $f^2(c) > 0$.

Now, f^2 is non-negative integrable on $[a, b]$ and c is a point of continuity of f^2 and $f^2(c) > 0$.

$\int_a^b f^2(x)\,dx > 0$, a contradiction to the hypothesis $\int_a^b f^2(x)\,dx = 0$

$\therefore f(c) = 0$.

Thus $f(x) = 0$ at every point of continuity $x \in [a, b]$.

Problem 31. The function $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both continuous on $[a, b]$ and $\int_a^b |f-g| = 0$. Prove that $f = g$. Give an example of functions f and g, both integrable on $[a, b]$ such that $\int_a^b |f-g| = 0$, but $f \neq g$.

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Solution : $f, g : [a, b] \to \mathbb{R}$ be the continuous on $[a, b]$

$\Rightarrow |f-g|$ is continuous on $[a, b]$

Now $|f-g|$ is non-negative continuous on $[a, b]$ and $\int_a^b |f-g| = 0$

$\Rightarrow |f-g|(x) = 0 \ \forall \ x \in [a, b]$

$\Rightarrow f(x) = g(x) \ \forall \ x \in [a, b]$

$\Rightarrow f = g.$

2nd part : Let : $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = 0, x \in [a, b]$ and $g : [a, b] \to \mathbb{R}$ is defined by

$$g(x) = 0, \ a < x \leq b$$
$$= 1, \ x = a$$

Then $|f(x) - g(x)| = 0, \ a < x \leq b$

$$= 1, \ x = a$$

f is continuous on $[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

g is bounded on $[a, b]$ and continuous on $[a, b]$ except at $x = a$. $\therefore$ g is integrable on $[a, b]$.

Now $|f-g| = f$ on $[a, b]$ except at $x = a$

$\Rightarrow \int_a^b |f-g| = \int_a^b f = 0$

Clearly, $f \neq g$ on $[a, b]$

Problem 32. A function f is continuous for $x \geq 0$ and $f(x) \neq 0$ for all $x > 0$. If $\{f(x)\}^2 = 2\int_0^x f(t)\,dt$ prove that $f(x) = x$ for all $x \geq 0$.

Solution : $\{f(x)\}^2 = 2\int_0^x f(t)\,dt, \ x \geq 0$ ... (1)

From (1), $\{f(0)\}^2 = 2\int_0^x f(t)\,dt = 0$

$\Rightarrow f(0) = 0$ ... (2)

Let $F(x) = \int_0^x f(t)\,dt, \ x \geq 0$. Since f is continuous $\forall \ x \geq 0$,

F is differentiable $\forall \ x \geq 0$ and $F'(x) = f(x), \ x \geq 0$

Differentiating (1) w.r.t x, we get

$2f(x) f'(x) = 2f(x), \ x \geq 0$

$\Rightarrow f'(x) = 1 \ \forall \ x > 0$, since $f(x) \neq 0 \ \forall \ x > 0$

Applying Lagranges mean value theorem on $[0, x]$ $(x > 0)$ to the function f, we get,

$f(x) - f(0) = f'(\xi)(x - 0)$ for some $\xi \in (0, x)$

$\Rightarrow f(x) = x$, since $f'(\xi) = 1, f(0) = 0$

$\therefore f(x) = x$ for $x > 0$ ... (3)

Combining (2) & (3), $f(x) = x, x \geq 0$. (Proved)

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Problem 33. Show that $-\frac{1}{2}<\int_0^1 \frac{x^3\cos 5x}{2+x^2}\,dx<\frac{1}{2}$.

Solution : We have, $\frac{x^3\cos 5x}{2+x^2}-x$

$$=\frac{-x^3(1-\cos 5x)-2x}{2+x^2}$$

$$=\frac{-2x\left[x^2\sin^2\frac{5x}{2}+1\right]}{2+x^2}$$

$$\le 0,\ \forall\, x\in[0,1]$$

$\therefore\ \frac{x^3\cos 5x}{2+x^2}\le x\ \forall\, x\in[0,1]$, equality holds when $x=0$

Again, $\frac{x^3\cos 5x}{2+x^2}+x=\frac{x^3(1+\cos 5x)+2x}{2+x^2}$

$$=\frac{2x\left[x^2\cos^2\frac{5x}{2}+1\right]}{2+x^2}\ge 0,\ \forall\, x\in[0,1]$$

$\therefore\ -x\le\frac{x^3\cos 5x}{2+x^2}\ \forall\, x\in[0,1]$, the equality holds when $x=0$

$\therefore\ -x\le\frac{x^3\cos 5x}{2+x^2}\le x\ \forall\, x\in[0,1]$ and equality holds when $x=0$

Let $h(x)=-x,\ x\in[0,1]$, $g(x)=x,\ x\in[0,1]$ and $f(x)=\frac{x^3\cos 5x}{2+x^2},\ x\in[0,1]$

$\therefore\ h(x)\le f(x)\le g(x)\ \forall\, x\in[0,1]$

Now f, g, h are all continuous on $[0,1]$ and hence integrable on $[0,1]$ and $h(x)<f(x)<g(x)\ \forall\, x\in(0,1]$

$$\Rightarrow\ \int_0^1 h(x)\,dx<\int_0^1 f(x)\,dx<\int_0^1 g(x)\,dx$$

$$\Rightarrow\ \int_0^1 -x\,dx<\int_0^1\frac{x^3\cos 5x}{2+x^2}\,dx<\int_0^1 x\,dx$$

$$\Rightarrow\ -\frac{1}{2}<\int_0^1\frac{x^3\cos 5x}{2+x^2}\,dx<\frac{1}{2}\quad\text{(Proved)}$$

Problem 34. Show that $\frac{\pi^3}{24\sqrt{2}}<\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin x+\cos x}\,dx<\frac{\pi^3}{24}$.

Solution : We have, $(\sin x+\cos x)^2=1+\sin 2x\ \forall\, x\in\mathbb{R}$.

Now, $0\le x\le\frac{\pi}{2}\Rightarrow 0\le\sin 2x\le 1$

$\Rightarrow 1\le 1+\sin 2x\le 2$

$\Rightarrow 1\le\sin x+\cos x\le\sqrt{2}$

$\therefore\ \frac{x^2}{\sqrt{2}}\le\frac{x^2}{\sin x+\cos x}\le x^2,\ \forall\, x\in\left[0,\frac{\pi}{2}\right]$

Let $h(x) = \dfrac{x^2}{\sqrt{2}}$, $x \in \left[0, \frac{\pi}{2}\right]$, $f(x) = \dfrac{x^2}{\sin x + \cos x}$, $x \in \left[0, \frac{\pi}{2}\right]$, $g(x) = x^2$, $x \in \left[0, \frac{\pi}{2}\right]$

Then f, g, h are all continuous on $\left[0, \frac{\pi}{2}\right]$ and hence integrable on $\left[0, \frac{\pi}{2}\right]$

Now $h(x) \le f(x) \le g(x)$, $\forall\, x \in \left[0, \frac{\pi}{2}\right]$ and $h\left(\frac{\pi}{6}\right) < f\left(\frac{\pi}{6}\right) < g\left(\frac{\pi}{6}\right)$

$$\therefore\ \int_0^{\frac{\pi}{2}} h(x)\,dx < \int_0^{\frac{\pi}{2}} f(x)\,dx < \int_0^{\frac{\pi}{2}} g(x)\,dx$$

$$\Rightarrow\ \int_0^{\frac{\pi}{2}} \frac{x^2}{\sqrt{2}}\,dx < \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x + \cos x}\,dx < \int_0^{\frac{\pi}{2}} x^2\,dx$$

$$\Rightarrow\ \frac{\pi^3}{24\sqrt{2}} < \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x + \cos x}\,dx < \frac{\pi^3}{24} \quad \text{(Proved)}$$

Problem 35. Show that, $\dfrac{\pi^3}{96} < \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2}{5+3\sin x}\,dx < \frac{\pi^3}{24}$

Solution : We have, $-1 \le \sin x \le 1\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow\ 5 - 3 \le 5 + 3\sin x \le 5 + 3\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow\ \frac{1}{8} \le \frac{1}{5+3\sin x} \le \frac{1}{2},\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow\ \frac{x^2}{8} \le \frac{x^2}{5+3\sin x} \le \frac{x^2}{2},\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Let $h(x) = \dfrac{x^2}{8}$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$f(x) = \dfrac{x^2}{5+3\sin x}$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $g(x) = \dfrac{x^2}{2}$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then h, f, g are all continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence integrable on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and

$h(x) \le f(x) \le g(x)\ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Also $h\left(\frac{\pi}{3}\right) < f\left(\frac{\pi}{3}\right) < g\left(\frac{\pi}{3}\right)$

$$\therefore\ \int_{-\pi/2}^{\pi/2} h(x)\,dx < \int_{-\pi/2}^{\pi/2} f(x)\,dx < \int_{-\pi/2}^{\pi/2} g(x)\,dx$$

$$\Rightarrow\ \int_{-\pi/2}^{\pi/2} \frac{x^2}{8}\,dx < \int_{-\pi/2}^{\pi/2} \frac{x^2\,dx}{5+3\sin x} < \int_{-\pi/2}^{\pi/2} \frac{1}{2}x^2\,dx$$

$$\Rightarrow\ \frac{1}{8}\left[\frac{x^3}{3}\right]_{-\pi/2}^{\pi/2} < \int_{-\pi/2}^{\pi/2} \frac{x^2\,dx}{5+3\sin x} < \frac{1}{2}\left[\frac{x^3}{3}\right]_{-\pi/2}^{\pi/2}$$

$$\Rightarrow\ \frac{\pi^3}{96} < \int_{-\pi/2}^{\pi/2} \frac{x^2\,dx}{5+3\sin x} < \frac{\pi^3}{24} \quad \text{(Proved)}$$

Problem 36. If α and β are positive acute angles, prove that

$$\beta < \int_0^{\beta} \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{\beta}{\sqrt{1=\sin^2\alpha\sin^2\beta}}.$$

Solution : Since sin x is monotonic increasing on $[0, \beta]$, $0 \le \sin x \le \sin \beta < 1\ \forall\ x \in [0, \beta] \subset \left[0, \frac{\pi}{2}\right)$

$\therefore\ 0 \le \sin^2 x \sin^2 \alpha \le \sin^2 \alpha \sin^2 \beta < 1$, since $0 < \sin \alpha < 1$

$\Rightarrow 1 \ge \sqrt{1-\sin^2 x \sin^2 \alpha} \ge \sqrt{1-\sin^2 \alpha \sin^2 \beta} > 0,\ \forall\ x \in [0, \beta]$

$\Rightarrow 1 \le \dfrac{1}{\sqrt{1-\sin^2 x \sin^2 \alpha}} \le \dfrac{1}{\sqrt{1-\sin^2 \alpha \sin^2 \beta}},\ \forall\ x \in [0, \beta]$

Let $h(x) = 1,\ x \in [0, \beta],\ f(x) = \dfrac{1}{\sqrt{1-\sin^2 x \sin^2 \alpha}},\ x \in [0, \beta]$

and $g(x) = \dfrac{1}{\sqrt{1-\sin^2 x \sin^2 \beta}},\ x \in [0, \beta]$

Then f, g, h are all continuous and hence integrable on $[0, \beta]$ and $h(x) < f(x) < g(x)$ $\forall\ x \in (0, \beta)$

$$\therefore\ \int_0^\beta h(x)\,dx < \int_0^\beta f(x)\,dx < \int_0^\beta g(x)\,dx$$

$$\Rightarrow\ \beta < \int_0^\beta \frac{dx}{\sqrt{1-\sin^2 \alpha \sin^2 x}} < \frac{\beta}{\sqrt{1-\sin^2 \alpha \sin^2 \beta}} \qquad \text{(Proved)}$$

Problem 37. Prove that $\dfrac{\pi}{6} < \displaystyle\int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} < \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}\ (k^2 < 1)$.

Solution : $0 \le x \le \frac{1}{2} \quad \Rightarrow 0 \le x^2 \le \frac{1}{4}$

$\Rightarrow -\frac{1}{4} \le -x^2 \le 0$

$\Rightarrow 1 - \frac{k^2}{4} \le 1 - k^2x^2 \le 1$

$\therefore\ 0 < (1 - x^2)\left(1-\frac{k^2}{4}\right) \le (1 - x^2)(1 - k^2x^2) \le 1 - x^2\ [\because\ 1 - x^2 > 0 \text{ for } 0 \le x \le \frac{1}{2}]$

$$\Rightarrow\ \frac{1}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{1}{\sqrt{1-\frac{k^2}{4}}\sqrt{1-x^2}}\ \forall\ x \in \left[0, \frac{1}{2}\right]$$

Let $f(x) = \dfrac{1}{\sqrt{(1-x^2)(1-k^2x^2)}},\ x \in \left[0, \frac{1}{2}\right]$

$h(x) = \dfrac{1}{\sqrt{1-x^2}},\ x \in \left[0, \frac{1}{2}\right]$ and $g(x) = \dfrac{1}{\sqrt{1-\frac{k^2}{4}}\sqrt{1-x^2}},\ x \in \left[0, \frac{1}{2}\right]$

Then $h(x) \le f(x) \le g(x)\ \forall\ x \in \left[0, \frac{1}{2}\right]$

Now f, g, h are all continuous on $\left[0, \frac{1}{2}\right]$ and so integrable on $\left[0, \frac{1}{2}\right]$

Further at $x = \frac{1}{3}$, $h\left(\frac{1}{3}\right) = \dfrac{1}{\sqrt{1-\frac{1}{9}}}$, $f\left(\frac{1}{3}\right) = \dfrac{1}{\sqrt{\left(1-\frac{1}{9}\right)\left(1-\frac{k^2}{9}\right)}}$

and $h\left(\frac{1}{3}\right) = \dfrac{1}{\sqrt{1-\frac{k^2}{4}}\sqrt{1-\frac{1}{9}}}$

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Now $\sqrt{1-\frac{1}{9}} > \sqrt{\left(1-\frac{1}{9}\right)\left(1-\frac{k^2}{9}\right)} > \sqrt{\left(1-\frac{1}{9}\right)\left(1-\frac{k^2}{4}\right)}$ $\left[\because 1 > 1-\frac{k^2}{9} > 1-\frac{k^2}{4}\right]$

$\Rightarrow h\left(\frac{1}{3}\right) < f\left(\frac{1}{3}\right) < g\left(\frac{1}{3}\right)$

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$$\therefore \int_0^{1/2} h(x)\,dx < \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} < \int_0^{1/2} g(x)\,dx$$

$$\Rightarrow \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} < \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} < \int_0^{1/2} \frac{dx}{\sqrt{1-\frac{k^2}{4}}\sqrt{1-x^2}}$$

$$\Rightarrow \left[\sin^{-1} x\right]_0^{1/2} < \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} < \frac{1}{\sqrt{1-\frac{k^2}{4}}}\left[\sin^{-1} x\right]_0^{1/2}$$

$$\Rightarrow \frac{\pi}{6} < \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} < \frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{k^2}{4}}} \quad \text{(Proved)}$$

2.6 Fundamentals Theorems :

Definition : Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then f is integrable on $[a, x]$ for every $x \in [a, b]$. The function $F : [a, b] \to \mathbb{R}$, defined by

$F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$ is called indefinite integral of f on $[a, b]$.

***Theorem 2.6.1* :** Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$. Then F is continuous on $[a, b]$.

Proof : $f \in \mathcal{R}[a, b]$

$\Rightarrow f$ is bounded on $[a, b]$

$\Rightarrow \exists\, B > 0$ such that $|f(t)| < B, \forall\, t \in [a, b]$

Let $x', x'' \in [a, b]$.

Then $F(x'') - F(x') = \int_a^{x''} f(t)\,dt - \int_a^{x'} f(t)\,dt = \int_{x'}^{x''} f(t)\,dt$

If $x' < x''$, then $|F(x'') - F(x')| = \left|\int_{x'}^{x''} f(t)\,dt\right|$

$$\le \int_{x'}^{x''} |f(t)|\,dt$$

$$\le B(x'' - x') \quad \text{... (i)}$$

If $x'' < x'$, then $|F(x'') - F(x')| = \left|\int_{x'}^{x''} f(t)\,dt\right|$

$$= \left|\int_{x''}^{x'} f(t)\,dt\right|$$

$$\le \int_{x''}^{x'} |f(t)|\,dt$$

$$\le B(x' - x'') \quad \text{... (ii)}$$

$\therefore$ From (i) and (ii), $\forall\, x', x'' \in [a, b]$, $|F(x'') - F(x')| \le B|x'' - x'|$

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Let $\varepsilon > 0$ be arbitrary. Let $\delta = \frac{\varepsilon}{B}$.

Then $x', x'' \in [a, b]$ and $|x'' - x'| < \delta \Rightarrow |F(x'') - F(x')| < \varepsilon$

$\Rightarrow F$ is uniformly continuous on $[a, b]$.

$\Rightarrow F$ is continuous on $[a, b]$.

Theorem 2.6.2 : Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$. If f is continuous on $[a, b]$ then F is differentiable on $[a, b]$ and $F'(x) = f(x)\ \forall\ x \in [a, b]$.

Proof : Let f be continuous on $[a, b]$.

Let $c \in [a, b)$. Let $\varepsilon > 0$ be arbitrary.

Then $\exists\ \delta > 0$ such that $f(c) - \varepsilon < f(x) < f(c) + \varepsilon\ \forall\ x \in [c, c + \delta) \subset [a, b]$

Let $0 < h < \delta$. Then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon\ \forall\ x \in [c, c + h] \subset [a, b]$

$$\Rightarrow \int_c^{c+h} \{f(c) - \varepsilon\}\,dx < \int_c^{c+h} f(x)\,dx < \int_c^{c+h} \{f(c) + \varepsilon\}\,dx$$

$$\Rightarrow (f(c) - \varepsilon)h < F(c + h) - F(c) < (f(c) + \varepsilon)h$$

$$\Rightarrow -\varepsilon < \frac{F(c+h) - F(c)}{h} - f(c) < \varepsilon$$

$\Rightarrow \left|\frac{F(c+h) - F(c)}{h} - f(c)\right| < \varepsilon$. This holds for all h satisfying $0 < h < \delta$.

$$\Rightarrow \lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$$

$$\Rightarrow R\,F'(c) = f(c) \qquad \text{... (i)}$$

Let $c \in (a, b]$. Let $\varepsilon > 0$ be arbitrary.

Then $\exists\ \eta > 0$ s.t. $f(c) - \varepsilon < f(x) < f(c) + \varepsilon\ \forall\ x \in (c - \eta, c] \subset [a, b]$

Let $0 < h < \eta$. Then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon\ \forall\ x \in [c - h, c] \subset [a, b]$

$$\Rightarrow (f(c) - \varepsilon)h < \int_{c-h}^{c} f(x)\,dx < (f(c) + \varepsilon)h$$

$$\Rightarrow \left|\frac{F(c) - F(c-h)}{h} - f(c)\right| < \varepsilon$$

$$\Rightarrow \left|\frac{F(c+h') - F(c)}{h'} - f(c)\right| < \varepsilon,\ h' = -h,\ -\eta < h' < 0$$

$$\Rightarrow \lim_{h' \to 0^-} \frac{F(c+h') - F(c)}{h'} = f(c)$$

$$\therefore L\,F'(c) = f(c) \qquad \text{... (ii)}$$

From (i) and (ii), F is differentiable at $c \in [a, b]$ and $F'(c) = f(c)$.

Since $c \in [a, b]$ is arbitrary, F is differentiable on $[a, b]$ and $F'(x) = f(x)$, $x \in [a, b]$.

Alternative proof :

Let $c \in [a, b]$.

Case 1. Let $c \in (a, b)$. Let us choose h such that $c + h \in [a, b]$

$$\therefore F(c + h) - F(c) = \int_c^{c+h} f(t)\,dt$$

Then $\phi'(x) = f(x), \ \forall \ x \neq 0$

Now $\lim\limits_{x\to 0} \dfrac{\phi(x)-\phi(0)}{x-0} = \lim\limits_{x\to 0} x \sin \dfrac{1}{x} = 0$

$\Rightarrow \ \phi'(x) = 0 = f(0)$

$\therefore \ \phi'(x) = f(x) \ \forall \ x \in [-1, 1]$

$\Rightarrow$ ϕ is a primitive of f on $[-1, 1]$

Exercise : Let $f: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $\phi : [a, b] \to \mathbb{R}$ be such that $\phi'(x) = f(x)$ $\forall \ x \in [a, b]$, then $\int_a^b f = \phi(b) - \phi(a)$.

Solution : $f: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$

$\Rightarrow$ F is an antiderivative of f on $[a, b]$, where $F(x) = \int_a^x f(t)\, dt$, $x \in [a, b]$

$\therefore \ F'(x) = \phi'(x) = f(x) \ \forall \ x \in [a, b]$

$\Rightarrow \ F(x) = \phi(x) + c, x \in [a, b]$, c is a real constant

$\therefore \ F(a) = \phi(a) + c$

$\Rightarrow \ c = -\phi(a)$, since $F(a) = 0$

$\therefore \ F(x) = \phi(x) - \phi(a), x \in [a, b]$

$\Rightarrow \ F(b) = \phi(b) - \phi(a)$

$\Rightarrow \ \int_a^b f = \phi(b) - \phi(a)$ (Proved)

***Theorem 2.6.3* : (Fundamental theorem of Integral calculus).**

If (i) $f: [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, and

(ii) f possesses a primitive ϕ on $[a, b]$, then $\int_a^b f = \phi(b) - \phi(a)$.

Proof : Let $P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\} \in \mathcal{P}[a, b]$

Let $M_r = \sup\limits_{x\in[x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x\in[x_{r-1}, x_r]} f(x)$, $r = 1, 2, \dots, n$

Now $\phi'(x) = f(x) \ \forall \ x \in [a, b]$

$\Rightarrow \ \phi'(x) = f(x) \ \forall \ x \in [x_{r-1}, x_r], \ \forall \ r = 1, 2, \dots, n$

$\therefore$ ϕ satisfies all conditions of LMVT on $[x_{r-1}, x_r]$ for $r = 1, 2, \dots, n$

$\therefore \ \phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r)\,(x_r - x_{r-1})$, for some $\xi_r \in (x_{r-1}, x_r)$, $r = 1, 2, \dots, n$

$\Rightarrow \ \sum\limits_{r=1}^{n} \{\phi(x_r) - \phi(x_{r-1})\} = \sum\limits_{r=1}^{n} f(\xi_r)\,(x_r - x_{r-1})$

$\Rightarrow \ \phi(b) - \phi(a) = \sum\limits_{r=1}^{n} f(\xi_r)\,(x_r - x_{r-1})$ ... (i)

Now $m_r \le f(\xi_r) \le M_r$ for $r = 1, 2, \dots, n$, since $\xi_r \in [x_{r-1}, x_r]$

$\Rightarrow \ \sum\limits_{r=1}^{n} m_r\,(x_r - x_{r-1}) \le \sum\limits_{r=1}^{n} f(\xi_r)\,(x_r - x_{r-1}) \le \sum\limits_{r=1}^{n} M_r\,(x_r - x_{r-1})$

(11)

$\Rightarrow L(P, f) \le \phi(b) - \phi(a) \le U(P, f)$, by (i)

This holds for all $P \in \mathcal{P}[a, b]$

$\therefore \sup_P L(P, f) \le \phi(b) - \phi(a) \le \inf_P U(P, f)$

$$\Rightarrow \underline{\int_a^b} f \le \phi(b) - \phi(a) \le \overline{\int_a^b} f$$

$$\Rightarrow \int_a^b f \le \phi(b) - \phi(a) \le \int_a^b f \text{, since } f \in \mathcal{R}[a, b],\ \underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$$

$$\Rightarrow \int_a^b f = \phi(b) - \phi(a) \quad \text{(Proved)} \qquad \text{... (ii)}$$

Remark: Since ϕ is differentiable on $[a, b]$, ϕ satisfies all conditions of LMVT on $[a, b]$

$\Rightarrow \exists\, \theta \in (0, 1)$ such that $\phi(b) - \phi(a) = \phi'(a + \theta(b - a)).(b - a)$

$= f(a + \theta(b - a))\ (b - a)$

$\therefore$ from (ii), $\int_a^b f = f(a + \theta(b - a))\ (b - a)$ for some θ satisfying $0 < \theta < 1$.

Exercise : Show by an example that an integrable function on $[a, b]$ may not have primitive on $[a, b]$.

Solution : Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$f(x) = -1, \quad -1 \le x \le 0$

$\quad\ = 1, \quad 0 \le x \le 1$

Then f is bounded on $[-1, 1]$ and continuous on $[-1, 1]$ except at $x = 0$. $\therefore f$ is integrable on $[-1, 1]$

If possible, let ϕ be a primitive of f on $[-1, 1]$

Then $\phi'(x) = f(x)\ \forall\ x \in [-1, 1]$

$\therefore \phi'(-1) = f(-1) = -1, \phi'(1) = f(1) = 1$

Since ϕ is differentiable on $[-1, 1]$ and $\phi'(-1) \ne \phi'(1)$, ϕ' must assume every value between $\phi'(-1)$ and $\phi'(1)$ i.e. f attains every value between -1 and 1, but this is not so.

$\therefore \phi$ does not exist i.e. f has no primitive on $[-1, 1]$.

Exercise : Show by an example that a non integrable function on $[a, b]$ may have a primitive on $[a, b]$.

Solution : Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, \quad x \ne 0$$
$$= 0 \qquad\qquad\qquad\qquad\qquad\quad, x = 0$$

f is unbounded in every neighbourhood of 0 and so f is not integrable on $[-1, 1]$

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by $\phi(x) = x^2 \sin\frac{1}{x^2}, x \ne 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = 0 \qquad\quad , x = 0$

Then $\phi'(x) = f(x)$ for $x \ne 0$

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Now $\lim_{x\to 0}\frac{\phi(x)-\phi(0)}{x-0}=\lim_{x\to 0} x^2 \sin\frac{1}{x^2}=0=f(0)$

$\Rightarrow \phi'(0)=0=f(0)$.

$\therefore \phi'(x)=f(x), \forall x \in [-1, 1]$

$\Rightarrow \phi$ is a primitive of f on $[-1, 1]$

Theorem 2.6.4 **:** If (i) $f:[a, b]\to \mathbb{R}$ be integrable on $[a, b]$ and (ii) $\exists\ \phi:[a, b]\to\mathbb{R}$ such that ϕ is continuous on $[a, b]$ and $\phi'(x)=f(x)\ \forall\ x\in(a, b)$, then $\int_a^b f=\phi(b)-\phi(a)$.

Proof : Let $P=\{a=x_0<x_1<x_2<\ldots<x_n=b\}\in\mathcal{P}[a, b]$

Let $M_r=\sup_{x\in[x_{r-1}, x_r]} f(x)$, $m_r=\inf_{x\in[x_{r-1}, x_r]} f(x)$, $r=1, 2, \ldots, n$

Now ϕ is continuous on $[x_{r-1}, x_r]$ and ϕ is differentiable on (x_{r-1}, x_r) for $r=1, 2, \ldots, n$

$\Rightarrow \phi$ satisfies all conditions of LMVT on $[x_{r-1}, x_r]$ $(r=1, 2, \ldots, n)$

$\Rightarrow \phi(x_r)-\phi(x_{r-1})=\phi'(\xi_r)(x_r-x_{r-1})$ for some $\xi_r\in(x_{r-1}, x_r)$, $r=1, 2, \ldots, n$

$=f(\xi_r)(x_r-x_{r-1})$

$$\Rightarrow \sum_{r=1}^{n}\{\phi(x_r)-\phi(x_{r-1})\}=\sum_{r=1}^{n} f(\xi_r)(x_r-x_{r-1})$$

$$\Rightarrow \phi(b)-\phi(a)=\sum_{r=1}^{n} f(\xi_r)(x_r-x_{r-1}) \quad \ldots \text{(i)}$$

Now $m_r\le f(\xi_r)\le M_r$, $r=1, 2, \ldots, n$

$$\Rightarrow \sum_{r=1}^{n} m_r(x_r-x_{r-1})\le\sum_{r=1}^{n} f(\xi_r)(x_r-x_{r-1})\le\sum_{r=1}^{n} M_r(x_r-x_{r-1})$$

$\Rightarrow L(P, f)\le\phi(b)-\phi(a)\le U(P, f)$, by (i)

This holds for all $P\in\mathcal{P}[a, b]$

$\therefore \sup_P L(P, f)\le\phi(b)-\phi(a)\le\inf_P U(P, f)$

$$\Rightarrow \underline{\int_a^b} f\le\phi(b)-\phi(a)\le\overline{\int_a^b} f$$

$\Rightarrow \int_a^b f\le\phi(b)-\phi(a)\le\int_a^b f$, since $f\in\mathcal{R}[a, b]$, $\underline{\int_a^b} f=\overline{\int_a^b} f=\int_a^b f$

$\Rightarrow \int_a^b f=\phi(b)-\phi(a)$. (Proved) primitive on [a, b].

******** For examination

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But continuity of f is only a sufficient condition for the existence of a primitive.

If the function be not continuous, then it can have a primitive –

for example,

Let $f : [-1, 1] \to R$ be defined by

$$f(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0.$$

Here f is not continuous at 0, since $\lim_{x\to 0} f(x) = \lim_{x \to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist.

Now let $\phi : [-1, 1] \to R$ be defined by

$$\phi(x) = x^2 \sin\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

When $x \neq 0$, $\phi'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$.

When $x = 0$,

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$$\lim_{x \to 0} \frac{\phi(x) - \phi(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x}}{x}$$

$$= \lim_{x \to 0} x \sin\frac{1}{x}$$

$$= 0, \quad \text{since } \lim_{x \to 0} x = 0 \text{ and } \sin\frac{1}{x} \text{ is bounded.}$$

Therefore, $\phi'(0) = 0$.

This shows that ϕ is a primitive of f.

Here f is not continuous on $[-1, 1]$, but f has a primitive on $[-1, 1]$.

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NO'CH2000

If $f : [a, b] \to R$ be continuous on $[a, b]$, and ϕ be a primitive of f on $[a, b]$, then $\int_a^b f = \phi(b) - \phi(a)$.

i.e. the evaluation of the integral $\int_a^b f$ can be done by the help of a primitive.

proof

Since f is continuous on $[a, b]$, f is integrable on $[a, b]$.

Let $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$.

Since f is continuous on $[a,b]$, $F'(x) = f(x)$, for all $x \in [a,b]$.

This shows that F is a primitive of f.

Therefore $F(x) = \phi(x) + c$, where c is a constant.

Now $F(a) = \int_a^a f(t)\,dt = 0$ (15)

$\therefore \phi(a) + c = 0$

or $c = -\phi(a)$

Now $\int_a^b f(t)\,dt = F(b)$

$= \phi(b) + c$

$= \phi(b) - \phi(a)$.

Fundamental theorem of integral calculus

Statement:- If $f : [a,b] \to R$ be integrable on $[a,b]$ and
CH-2001
if f possesses a primitive ϕ on $[a,b]$,

then $\int_a^b f = \phi(b) - \phi(a)$.

Proof - 2000, NO.

****** Give an example of a function f such that f is integrable on $[a,b]$, but f has no primitive on $[a,b]$.

$\Rightarrow$ Let us consider the function f defined on $[-1,1]$ by

$f(x) = \operatorname{sgn} x$, $x \in [-1,1]$.

$f(x) = 1$ when $0 < x \leq 1$

$= 0$ when $x = 0$

$= -1$ when $-1 \leq x < 0$

f is continuous on $[-1,1]$ except at 0.

Since the number of points of discontinuity of f is finite, f is integrable on $[-1,1]$.

We now prove that f has no primitive on $[-1,1]$.

Let g be a primitive of f on $[-1,1]$.

Then $g'(x) = f(x) \quad \forall x \in [-1,1]$.

f has a jumped discontinuity at 0.

Therefore g' must have a jumped discontinuity at 0.

But "A derived function can not have a jumped discontinuity".

Therefore g does not exist.

In other words f has no primitive.

******C.H. 2001 **

CH-89 Give an example of a function f such that f has a primitive on a closed interval $[-1,1]$. But f is not integrable on $[-1,1]$.

⇒ ~~Let~~ Consider the function f defined on

[-1,1] by

$f(x) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, \quad x \neq 0$

$= 0, \quad x = 0$

$\phi(x) = x^2 \sin\frac{1}{x^2}$

$\phi'(x) = 2x\sin\frac{1}{x^2} + x^2\cos(\frac{1}{x^2})(\frac{-2}{x^3})$

$= 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$

Let us consider the function ϕ defined by

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$$\phi(x) = x^2 \sin\frac{1}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

When $x \neq 0$, $\phi'(x) = 2x\sin\frac{1}{x^2} + x^2\cos(\frac{1}{x^2})(-\frac{2}{x^3})$

$$= 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$$

$$\lim_{x\to 0}\frac{\phi(x)-\phi(0)}{x-0} = \lim_{x\to 0}\frac{x^2\sin\frac{1}{x^2} - 0}{x-0} = 0,$$ since $\lim_{x\to 0} x = 0$ and $|\sin\frac{1}{x^2}| \leq 1$

$\therefore \phi'(0) = 0$.

This shows that ϕ is a primitive of f on $[-1,1]$.

[$\cos\frac{1}{x^2} = 2n\pi$, $x = 0 \Rightarrow \infty$]

Here f is unbounded on $[-1,1]$.

Therefore f is not integrable on $[-1,1]$.

CH'98 f is defined on $[1,3]$ by

$$f(x) = 1, \quad 1 \leq x < 2$$
$$= 2, \quad 2 \leq x \leq 3.$$

State, with reasons —

i) whether $\int_1^3 f(x)dx$ exists.

ii) whether the fundamental theorem of

integral calculus is applicable to $f(x)$ in [1,3]

⇒ (i) f is continuous on [1, 3] except at 2.

Since f is continuous on [1, 3] except at a finite number of points, f is integrable on [1, 3]. (18)

(ii) We now show that f has no primitive on [1,3].

Let g be a primitive of f on [1, 3].

Then $g'(x) = f(x)$ on [1, 3].

Here f has a jumped discontinuity at 2. Therefore g' must have a jumped discontinuity at 2.

But 'A derived function can not have a jumped discontinuity.

Therefore g does not exist.

Therefore, the evaluation of $\int_1^3 f(x)\,dx$ can not be done by the fundamental theorem of integral calculus.

(iii) This result holds ...