E - Learning Materials for Sem - 4

Unit - 1 , CC - 8

Topic - Riemann Integration

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Since f is bounded on $[a, b]$, $\exists\ \delta_1 > 0, \delta_2 > 0$ such that

$U(P, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}\ \forall\ P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta_1$

and $L(P, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}\ \forall\ P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta_2$ [Darboux theorem]

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $U(P, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$ and $L(P, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}\ \forall\ P \in \mathcal{P}[a, b]$ satisfying $\|P\|$

$\Rightarrow U(P, f) - L(P, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2} - \underline{\int_a^b} f + \frac{\varepsilon}{2} = \varepsilon$, by (1), $\forall\ P \in \mathcal{P}[a, b]$ satisfying $\|P\|$

The condition is sufficient

Let the condition be satisfied.

Let $\varepsilon > 0$ be arbitrary. Then by the conditon $\exists\ \delta > 0$ such that

$U(P, f) - L(P, f) < \varepsilon\ \forall\ P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

Now $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P, f) - L(P, f)$ holds for all $P \in \mathcal{P}[a, b]$

$\Rightarrow 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P, f) - L(P, f) < \varepsilon\ \forall\ P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$

$\Rightarrow 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon$

$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f$, since $\varepsilon > 0$ is arbitrary.

$\Rightarrow f \in \mathcal{R}[a, b]$

This completes the proof.

SOLVED PROBLEMS

Problem 4. Let $f: [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] - \mathbb{Q} \end{cases}$.

Examine whether f is integrable on [0, 1].

Solution : $f(x) = \begin{cases} x, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] - \mathbb{Q} \end{cases}$

Since $|f(x)| \le 1, \forall\ x \in [0, 1]$, f is bounded on [0, 1]

Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\} \in \mathcal{P}[0, 1]$

Let $M_r = \sup\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $m_r = \inf\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $r = 1, 2, \ldots, n$

②

Then $M_r = f\left(\frac{r}{n}\right) = \frac{r}{n}$, $m_r = 0$, $r = 1, 2, ..., n$

$$\therefore\ U(P_n, f) = \sum_{r=1}^{n} M_r \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$L(P_n, f) = \sum_{r=1}^{n} m_r \cdot \frac{1}{n} = 0$$

Now $\{P_n\}_n$ is a sequence of partitions of [0, 1] such that

$$\| P_n \| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$\therefore\ \overline{\int_0^1} f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} \frac{1}{2}\left(n+\frac{1}{n}\right) = \frac{1}{2}$$

and $\underline{\int_0^1} f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} 0 = 0$

Since, $\underline{\int_0^1} f \neq \overline{\int_0^1} f$, $f \notin \mathcal{R}[0, 1]$

Problem 5. Let f: [0, 1] → $\mathbb{R}$ be defined by $f(x) = \begin{cases} x^2, & x \in [0,1] \cap \mathbb{Q} \\ x^3, & x \in [0,1] - \mathbb{Q} \end{cases}$

Examine if f is Riemann integrable on [0, 1].

Solution : $f(x) = \begin{cases} x^2, & x \in [0,1] \cap \mathbb{Q} \\ x^3, & x \in [0,1] - \mathbb{Q} \end{cases}$

Since $\forall\, x \in [0, 1]$, $|f(x)| \leq 1$, f is bounded on [0, 1].

Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\right\}$ be a partition of [0, 1].

Then $\{P_n\}_n$ is a sequence of partitions of [0, 1] such that

$$\| P_n \| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

Let $M_r = \sup\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $m_r = \inf\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $r = 1, 2, ..., n$

$f\big|_{\left[\frac{r-1}{n}, \frac{r}{n}\right] \cap \mathbb{Q}}$ is monotonic increasing on $\left[\frac{r-1}{n}, \frac{r}{n}\right] \cap \mathbb{Q}$

$$\therefore\ \sup\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] \cap \mathbb{Q}\right\} = f\left(\frac{r}{n}\right) = \left(\frac{r}{n}\right)^2$$

and $\inf\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] \cap \mathbb{Q}\right\} = f\left(\frac{r-1}{n}\right) = \left(\frac{r-1}{n}\right)^2$

$f\big|_{\left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}}$ is monotonic increasing on $\left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}$

$\therefore \sup\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}\right\} = \lim\limits_{m\to\infty} f(\alpha_m)$, where $\{\alpha_m\}_m$ is a sequence of points in $\left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}$ converges to $\frac{r}{n}$

and $\inf\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}\right\} = \lim\limits_{m\to\infty} f(\beta_m)$, where $\{\beta_m\}_m$ is a sequence of points in $\left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}$ converges to $\frac{r-1}{n}$

Now $\alpha_m \to \frac{r}{n}$ as $m \to \infty \Rightarrow f(\alpha_m) = \alpha_m^3 \to \left(\frac{r}{n}\right)^3$ as $m \to \infty$

and $\beta_m \to \frac{r-1}{n}$ as $m \to \infty \Rightarrow f(\beta_m) = \beta_m^3 \to \left(\frac{r-1}{n}\right)^3$ as $m \to \infty$

$\therefore \sup\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}\right\} = \left(\frac{r}{n}\right)^3$

and $\inf\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}\right\} = \left(\frac{r-1}{n}\right)^3$

Now $x^3 \le x^2 \; \forall\, x \in [0, 1]$

$\Rightarrow M_r = \left(\frac{r}{n}\right)^2$ and $m_r = \left(\frac{r-1}{n}\right)^3$, $r = 1, 2, \ldots, n$

$$\therefore U(P_n, f) = \sum_{r=1}^{n} M_r \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{r=1}^{n} r^2 = \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$L(P_n, f) = \sum_{r=1}^{n} m_r \cdot \frac{1}{n} = \frac{1}{n^4}\sum_{r=1}^{n}(r-1)^3 = \frac{1}{n^4}\left\{\frac{(n-1)n}{2}\right\}^2 = \frac{1}{4}\left(1-\frac{1}{n}\right)$$

$$\therefore \overline{\int_0^1} f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}$$

$$\underline{\int_0^1} f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right) = \frac{1}{4}$$

Since $\underline{\int_0^1} f \ne \overline{\int_0^1} f$, $f \notin \mathcal{R}[0, 1]$

Problem 6. A function f is defined on $\left[0, \frac{\pi}{2}\right]$ by $f(x) = \begin{cases} \sin x, & x \text{ is rational} \\ x, & x \text{ is irrational} \end{cases}$

Evaluate $\overline{\int_0^{\pi/2}} f$, $\underline{\int_0^{\pi/2}} f$. Show that $f \notin \mathcal{R}\left[0, \frac{\pi}{2}\right]$.

Solution : $f(x) = \begin{cases} \sin x, & x \in \left[0, \frac{\pi}{2}\right] \cap \mathbb{Q} \\ x, & x \in \left[0, \frac{\pi}{2}\right] - \mathbb{Q} \end{cases}$

$|f(x)| \le \frac{\pi}{2} \; \forall\, x \in \left[0, \frac{\pi}{2}\right] \Rightarrow f$ is bounded on $\left[0, \frac{\pi}{2}\right]$.

Also $\sin x \le x \; \forall\, x \in \left[0, \frac{\pi}{2}\right]$, equality holds at $x = 0$

(4)

Let $P_n = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{n\pi}{2n}\right\}$ be a partition of $\left[0, \frac{\pi}{2}\right]$

Then $\| P_n \| = \frac{\pi}{2n} \to 0$ as $n \to \infty$

Let $I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$, $r = 1, 2, \ldots, n$

Let $M_r = \sup\limits_{x \in I_r} f(x)$, $m_r = \inf\limits_{x \in I_r} f(x)$, $r = 1, 2, \ldots, n$

Now $f\big|_{I_r \cap \mathbb{Q}}$ is monotonic increasing on $I_r \cap \mathbb{Q}$

$\therefore$ $\sup\{f(x) : I_r \cap \mathbb{Q}\} = \lim\limits_{m\to\infty} f(\alpha_m)$, where $\{\alpha_m\}_m$ is a sequence of points in $I_r \cap \mathbb{Q}$ and $\alpha_m \to \frac{r\pi}{2n}$ as $m \to \infty$

$$= \lim_{m\to\infty} \sin \alpha_m$$

$$= \sin \frac{r\pi}{2n}, \text{ since } \sin x \text{ is continuous on } \mathbb{R}.$$

Similarly, $\inf\{f(x) : x \in I_r \cap \mathbb{Q}\} = \sin \frac{(r-1)\pi}{2n}$

Again, $f\big|_{I_r - \mathbb{Q}}$ is monotonic increasing on $I_r - \mathbb{Q}$

is $\sup\{f(x) : x \in I_r - \mathbb{Q}\} = f\left(\frac{r\pi}{2n}\right) = \frac{r\pi}{2n}$

and $\inf\{f(x) : x \in I_r - \mathbb{Q}\} = f\left(\frac{(r-1)\pi}{2n}\right) = \frac{(r-1)\pi}{2n}$

$\therefore$ $M_r = \frac{r\pi}{2n}$, $m_r = \sin \frac{(r-1)\pi}{2n}$, $r = 1, 2, \ldots, n$

$$\therefore\ U(P_n, f) = \sum_{r=1}^{n} M_r \cdot \frac{\pi}{2n} = \frac{\pi^2}{4n^2}\sum_{r=1}^{n} r = \frac{\pi^2}{4n^2}\,\frac{n(n+1)}{2} = \frac{\pi^2}{8}\left(1+\frac{1}{n}\right)$$

$$L(P_n, f) = \sum_{r=1}^{n} m_r \cdot \frac{\pi}{2n} = \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \cdots + \sin\frac{(n-1)\pi}{2n}\right]$$

$$= \frac{\pi}{2n}\,\frac{\sin\frac{n}{2}\cdot\frac{\pi}{2n}\cdot\sin\frac{n-1}{2}\cdot\frac{\pi}{2n}}{\sin\frac{1}{2}\cdot\frac{\pi}{2n}}$$

$$\left[\sin x + \sin 2x + \cdots + \sin nx = \frac{\sin\frac{n+1}{2}x\,\sin\frac{n}{2}x}{\sin\frac{1}{2}x}\right]$$

$$= 2\cdot\frac{\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}\sin\frac{\pi}{4}\sin\left(1-\frac{1}{n}\right)\frac{\pi}{4}$$

$$\therefore\ \overline{\int_0^{\pi/2}} f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty}\frac{\pi^2}{8}\left(1+\frac{1}{n}\right) = \frac{\pi^2}{8}$$

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and $\underline{\int_0^{\pi/2}} f = \lim_{n\to\infty} L(P_n, f) = 2\lim_{n\to\infty} \dfrac{\frac{\pi}{4n}}{\sin\frac{\pi}{4n}} \sin\frac{\pi}{4}\sin\left(1-\frac{1}{n}\right)\frac{\pi}{4}$

$= 2\sin^2\frac{\pi}{4}$

$= 1$

Since $\overline{\int_0^{\pi/2}} f \neq \underline{\int_0^{\pi/2}} f$, $f \notin \mathcal{R}\left[0, \frac{\pi}{2}\right]$

Problem 7. A function f is defined on [0, 1] by $f(x) = \begin{cases} x^2 + x^3, & x \text{ is rational} \\ x + x^2, & x \text{ is irrational}\end{cases}$

Evaluate $\underline{\int_0^1} f$, $\overline{\int_0^1} f$. show that $f \notin \mathcal{R}[0, 1]$.

Solution : $f(x) = \begin{cases} x^2 + x^3, & x \in [0,1] \cap \mathbb{Q} \\ x + x^2, & x \in [0,1] - \mathbb{Q}\end{cases}$

$|f(x)| \leq 2\ \forall\ x \in [0, 1] \Rightarrow f$ is bounded on [0, 1].

We note that, $(x + x^2) - (x^2 + x^3) = x - x^3 \geq 0\ \forall\ x \in [0, 1]$

$\Rightarrow x + x^2 \geq x^2 + x^3\ \forall\ x \in [0, 1]$

Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}$ be a partition of [0, 1]

Then $\|P_n\| = \frac{1}{n} \to 0$ as $n \to \infty$

Let $M_r = \sup\limits_{x\in\left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $m_r = \inf\limits_{x\in\left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $r = 1, 2, \ldots, n$

Now $f\Big|_{\left[\frac{r-1}{n}, \frac{r}{n}\right]\cap\mathbb{Q}}$ is monotonic increasing on $\left[\frac{r-1}{n}, \frac{r}{n}\right]\cap\mathbb{Q}$

$\therefore\ \sup\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]\cap\mathbb{Q}\right\} = f\left(\frac{r}{n}\right) = \left(\frac{r}{n}\right)^2 + \left(\frac{r}{n}\right)^3$

$\inf\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]\cap\mathbb{Q}\right\} = f\left(\frac{r-1}{n}\right) = \left(\frac{r-1}{n}\right)^2 + \left(\frac{r-1}{n}\right)^3$

Again $f\Big|_{\left[\frac{r-1}{n}, \frac{r}{n}\right]-\mathbb{Q}}$ is monotonic increasing on $\left[\frac{r-1}{n}, \frac{r}{n}\right]$

$\therefore\ \sup\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]-\mathbb{Q}\right\} = \lim\limits_{m\to\infty} f(\alpha_m)$, where $\{\alpha_m\}_m$ is a sequence i[n] $\left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}$ converging to $\frac{r}{n}$

$= \lim\limits_{m\to\infty}(\alpha_m + \alpha_m^2)$

$= \frac{r}{n} + \left(\frac{r}{n}\right)^2$, since $\alpha_m \to \frac{r}{n}$ as $m \to \alpha$

$\inf\left\{f(x) : x \in \left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}\right\} = \lim_{m\to\infty} f(\beta_m)$, where $\{\beta_m\}_m$ is a sequence in $\left[\frac{r-1}{n}, \frac{r}{n}\right] - \mathbb{Q}$ converging to $\frac{r-1}{n}$

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$$= \lim_{m\to\infty} (\beta_m + \beta_m^2)$$

$$= \frac{r-1}{n} + \left(\frac{r-1}{n}\right)^2, \text{ since } \alpha_m \to \frac{r-1}{n} \text{ as } m \to \alpha$$

$$\therefore\ M_r = \frac{r}{n} + \left(\frac{r}{n}\right)^2,\ m_r = \left(\frac{r-1}{n}\right)^2 + \left(\frac{r-1}{n}\right)^3, r = 1, 2, \ldots, n$$

$$\therefore\ U(P_n, f) = \sum_{r=1}^{n} M_r \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n} r + \frac{1}{n^3}\sum_{r=1}^{n} r^2$$

$$= \frac{1}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$L(P_n, f) = \sum_{r=1}^{n} m_r \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{r=1}^{n}(r-1)^2 + \frac{1}{n^4}\sum_{r=1}^{n}(r-1)^3$$

$$= \frac{1}{n^3}\frac{(n-1)n(2n-1)}{6} + \frac{1}{n^4}\left\{\frac{(n-1)n}{2}\right\}^2$$

$$= \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{1}{4}\left(1-\frac{1}{n}\right)^2$$

$$\therefore\ \overline{\int_0^1} f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty}\left[\frac{1}{2}\left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\underline{\int_0^1} f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty}\left[\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{1}{4}\left(1-\frac{1}{n}\right)^2\right]$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$, $f \notin \mathcal{R}[0, 1]$

Problem 8. A function f defined on $\left[0, \frac{\pi}{4}\right]$ by $f(x) = \begin{cases} \cos x, & x \in \left[0, \frac{\pi}{4}\right] \cap \mathbb{Q} \\ \sin x, & x \in \left[0, \frac{\pi}{4}\right] - \mathbb{Q} \end{cases}$

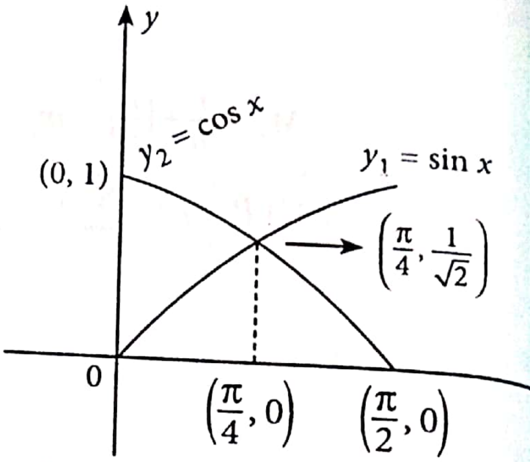
Evaluate $\overline{\int_0^{\pi/4}} f$ and $\underline{\int_0^{\pi/4}} f$ and show that $f \notin \mathcal{R}\left[0, \frac{\pi}{4}\right]$

Solution : $f(x) = \begin{cases} \cos x, & x \in \left[0, \frac{\pi}{4}\right] \cap \mathbb{Q} \\ \sin x, & x \in \left[0, \frac{\pi}{4}\right] - \mathbb{Q} \end{cases}$

Since $|f(x)| \leq 1 \ \forall\ x \in \left[0, \frac{\pi}{4}\right]$, f is bounded on $\left[0, \frac{\pi}{4}\right]$.

We note that $\sin x \leq \cos x \ \forall\ x \in \left[0, \frac{\pi}{4}\right]$, where equality holds at $x = \frac{\pi}{4}$, $y_1 =$ is increasing on $\left[0, \frac{\pi}{4}\right]$ while $y_2 = \cos x$ is decreasing on $\left[0, \frac{\pi}{4}\right]$

Let $P_n = \left(0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots, \frac{(n-1)\pi}{4n}, \frac{\pi}{4}\right)$ be a partition of $\left[0, \frac{\pi}{4}\right]$



Then $\| P_n \| = \frac{\pi}{4n} \to 0$ as $n \to \infty$

Let $I_r = \left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$, $r = 1, 2, \ldots, n$.

Let $M_r = \sup \{f(x) : x \in I_r\}$, $m_r = \inf \{f(x) : x \in I_r\}$, $r = 1, 2, \ldots, n$.

Now $f\big|_{I_r \cap \mathbb{Q}}$ is monotone decreasing on $I_r \cap \mathbb{Q}$

$\therefore$ $\sup \{f(x) : x \in I_r \cap \mathbb{Q}\} = \lim\limits_{m\to\infty} f(\alpha_m)$, where $\{\alpha_m\}_m$ is a sequence in $I_r \cap$ converging to $\frac{(r-1)\pi}{4n}$

$$= \lim_{m\to\infty} \cos \alpha_m$$

$$= \cos \frac{(r-1)\pi}{4n}, \quad \text{since } \cos x \text{ is continuous on } \mathbb{R}$$

and $\alpha_m \to \frac{(r-1)\pi}{4n}$ as $m \to \infty$

and $\inf \{f(x) : x \in I_r \cap \mathbb{Q}\} = \lim\limits_{m\to\infty} f(\beta_m)$, where $\{\beta_m\}_m$ is a sequence in $I_r \cap$ converging to $\frac{r\pi}{4n}$

$$= \lim_{m\to\infty} \cos \beta_m$$

$$= \cos \frac{r\pi}{4n}$$

Again, $f\big|_{I_r - Q}$ is monotone increasing on $I_r - \mathbb{Q}$.

$\therefore$ $\sup \{f(x) : x \in I_r - \mathbb{Q}\} = f\left(\frac{r\pi}{4n}\right) = \sin \frac{r\pi}{4n}$

and $\inf \{f(x) : x \in I_r - \mathbb{Q}\} = f\left(\frac{(r-1)\pi}{4n}\right) = \sin \frac{(r-1)\pi}{4n}$

$\therefore$ $M_r = \cos \frac{(r-1)\pi}{4n}$, $m_r = \sin \frac{(r-1)\pi}{4n}$, $r = 1, 2, \ldots, n$

$\therefore$ $U(P_n, f) = \sum\limits_{r=1}^{n} M_r \cdot \frac{\pi}{4n}$

(8)

$$= \frac{\pi}{4n}\sum_{r=1}^{n}\cos\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[1+\cos\frac{\pi}{4n}+\cos\frac{2\pi}{4n}+\cdots+\cos\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}+\frac{\pi}{4n}\,\frac{\cos\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)\cdot\sin\left(\frac{n-1}{2}\,\frac{\pi}{4}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{4n}+\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\cos\frac{\pi}{8}\sin\frac{\pi}{8}\left(1-\frac{1}{n}\right)$$

$$\therefore \overline{\int_0^{\pi/4}} f = \lim_{n\to\infty} U(P_n, f)$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}+\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\cos\frac{\pi}{8}\sin\frac{\pi}{8}\left(1-\frac{1}{n}\right)\right]$$

$$= 0 + 1.2\cos\frac{\pi}{8}\ \sin\frac{\pi}{8}$$

$$= \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}$$

$$L(P_n, f) = \sum_{r=1}^{n} m_r\cdot\frac{\pi}{4n}$$

$$= \frac{\pi}{4n}\sum_{r=1}^{n}\sin\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\sin\frac{\pi}{4n}+\sin\frac{2\pi}{4n}+\cdots+\sin\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}\,\frac{\sin\frac{n\pi}{8n}\sin\frac{(n-1)\pi}{8n}}{\sin\frac{\pi}{8n}}$$

$$= \frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\sin\frac{\pi}{8n}\sin\left(1-\frac{1}{n}\right)\frac{\pi}{8}$$

$$\therefore \underline{\int_0^{\pi/4}} f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty}\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\sin\frac{\pi}{8n}\sin\left(1-\frac{1}{n}\right)\frac{\pi}{8}$$

$$= 2\sin^2\frac{\pi}{8}$$

$$= 1-\cos\frac{\pi}{4} = 1-\frac{1}{\sqrt{2}}$$

Since $\underline{\int_0^{\pi/4}} f \neq \overline{\int_0^{\pi/4}} f,\ f \notin \mathcal{R}\left[0, \frac{\pi}{4}\right]$ (proved)

Problem 9. Let $f: [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ $(0 < a < b)$. Let P_n $(a, ar, ar^2, ..., ar^n)$ whe $r^n = \frac{b}{a}$. Consider the sequence $\{P_n\}_n$ of partitions of $[a, b]$, prove that $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$ and $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$.

9

Evaluate $\underline{\int_1^2} f$ and $\overline{\int}$ when (i) $f(x) = x^9, x \in [1, 2]$, (ii) $f(x) = x^{99}, x \in [1, 2]$

Solution : $f: [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ $(0 < a < b)$

$P_n = (a, ar, ar^2, ..., ar^n)$, where $r^n = \frac{b}{a}$.

Then $\| P_n \| = ar^{n-1}(r-1) = a\left(\frac{b}{a}\right)^{\frac{n-1}{n}}\left[\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1\right] \to 0$ as $n \to \infty$

Let $\varepsilon > 0$ be arbitrary.

Now f is bounded on $[a, b]$

$\Rightarrow \exists\, \delta > 0$ such that $U(P, f) < \overline{\int_a^b} f + \varepsilon\ \forall\, P \in \mathcal{P}[a, b]$ with $\| P \| < \delta$ ... (1

[Darboux theorem]

Again $\lim_{n\to\infty} \| P_n \| = 0 \Rightarrow \exists\, k \in \mathbb{N}$ such that $\| P_n \| < \delta\ \forall\, n \geq k$ ... (2

$\therefore$ by (1) & (2), $U(P_n, f) < \overline{\int_a^b} f + \varepsilon\ \forall\, n \geq k$

$$\Rightarrow \left| U(P_n, f) - \overline{\int_a^b} f \right| = U(P_n, f) - \overline{\int_a^b} f < \varepsilon,\ \forall\, n \geq k$$

$$\Rightarrow \lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$$

Similarly, $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$ (verify !)

(i) $f(x) = x^9, x \in [1, 2]$

$P_n = (1, r, r^2, ..., r^n)$, where $r^n = 2$

Then $\| P_n \| = r^{n-1}(r-1) = 2^{\frac{n-1}{n}}\left(2^{\frac{1}{n}} - 1\right) \to 0$ as $n \to \infty$

Let $M_s = \sup_{x \in [r^{s-1}, r^s]} f(x)$, $m_s = \inf_{x \in [r^{s-1}, r^s]} f(x)$, $s = 1, 2, ..., n$

Then $M_s = f(r^s) = (r^s)^9$ and $m_s = f(r^{s-1}) = r^{9(s-1)}$, $s = 1, 2, ..., n$

[$\because f$ is monotone increasing on [1, 2]

$$\therefore U(P_n, f) = \sum_{s=1}^{n} M_s\, r^{s-1}(r-1)$$

$$= \sum_{s=1}^{n} r^{9s} . r^{s-1}(r-1)$$

Now

$$U(P,f) = U(P_1,f) + U(P_2,f)$$

or $L(P,f) = L(P_1,f) + L(P_2,f)$

Therefore $U(P,f) - L(P,f)$

$$= [U(P_1,f) - L(P_1,f)] + [U(P_2,f) - L(P_2,f)]$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon.$$

This proves that f is integrable on $[a,b]$.

2nd part

Same as before.

Th^m

C.H '96

Let $\phi : [a,b] \to R$ be integrable on $[a,b]$. Let $f : [a,b] \to R$ be bounded on $[a,b]$ and $f(x) = \phi(x)$ $\forall x \in [a,b]$ except at a finite number of points $\in [a,b]$.
Then f is integrable on $[a,b]$ and $\int_a^b f = \int_a^b \phi$.

C.H. 2006 ***

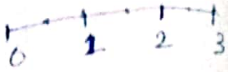
Let $f : [0,3] \to R$ be defined by $f(x) = [x]$, $x \in [0,3]$. Prove that f is integrable on $[0,3]$. Evaluate $\int_0^3 f$.

Ans.

$$f(x) = 0 \quad \text{when } 0 \le x < 1$$
$$= 1 \quad \text{when } 1 \le x < 2$$
$$= 2 \quad \text{when } 2 \le x < 3$$
$$= 3 \quad \text{when } x = 3.$$

f is continuous on $[0, 3]$ exc[illegible]
1, 2, 3.

Since the number of points of discontinuity is finite and f is bounded on $[0, 3]$, f is integrable on $[0, 3]$.



2nd Part :

$$\int_0^3 f = \int_0^1 f + \int_1^2 f + \int_2^3 f.$$

Let us define a function ϕ_1 on $[0, 1]$ by

$\phi_1(x) = 0$, $x \in [0, 1]$.

Then ϕ_1 is integrable on $[0, 1]$ and $\int_0^1 \phi_1 = 0(1-0)$
$= 0$

f is bounded on $[0, 1]$ and $f = \phi_1$ on $[0, 1]$ except at 1, therefore $\int_0^1 f = \int_0^1 \phi_1 = 0$.

Let us define a function ϕ_2 on $[1, 2]$ by $\phi_2(x) = 1$, $x \in [1, 2]$.

Then ϕ_2 is integrable on $[1, 2]$ and $\int_1^2 \phi_2 = 1(2-1)$
$= 1$.

f is bounded on $[1, 2]$ and $f = \phi_2$ on $[1, 2]$ except at 2, therefore $\int_1^2 f = \int_1^2 \phi_2 = 1$.

Let us define a function ϕ_3 on $[2, 3]$ by $\phi_3(x) = 2$, $x \in [2, 3]$.

Then ϕ_3 is integrable on $[2, 3]$ and $\int_2^3 \phi_3 = 2\cdot(3-2)$
$= 2$

f is bounded on $[2, 3]$ and $f = \phi_3$ except at 3, Therefore $\int_2^3 f = \int_2^3 \phi_3 = 2$.

(12)

$\therefore \int_0^3 f = 0 + 1 + 2 = 3.$ Ans

C.H. 2001 *** CH-96

With justification give an example of a function f defined on $[0,1]$, having infinite number of points of discontinuity but belongs to $R[0,1]$)

$f(x) = \frac{1}{n}, \quad \frac{1}{n+1} < x \leq \frac{1}{n} \quad (n = 1, 2, 3, \ldots)$

[illegible]

$= 0, \quad x = 0$

[illegible]

Prove that f is integrable on $[0,1]$. Evaluate $\int_0^1 f$.

Ans:

$f(x) = 1, \quad \frac{1}{2} < x \leq 1$

$= \frac{1}{2}, \quad \frac{1}{3} < x \leq \frac{1}{2}$

$= \frac{1}{3}, \quad \frac{1}{4} < x \leq \frac{1}{3}$

$= \ldots \quad \ldots \quad \ldots$

$= 0, \quad x = 0.$

f is bounded on $[0,1]$ and is continuous on $[0,1]$ except at $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

The set of points of discontinuity of f on $[0,1]$ is an infinite set having only one limit point. Therefore f is integrable on $[0,1]$.

2nd part: Let us define a function ϕ_1 on $[\frac{1}{2}, 1]$

by $\phi_1(x) = 1$, $x \in [\frac{1}{2}, 1]$.

Then ϕ_1 is continuous on $[\frac{1}{2}, 1]$. f is bounded on $[\frac{1}{2}, 1]$

(13)

~~Therefore~~ and $f = \phi_1$ except at $\frac{1}{2}$

$$\therefore \int_{1/2}^{1} f = \int_{1/2}^{1} \phi_1 = 1(1-\tfrac{1}{2}).$$

If $f(x) = k$ on $[a,b]$
then $\int_a^b f = k(b-a)$

Let us define a function ϕ_2 on $[\frac{1}{3}, \frac{1}{2}]$ by. $\phi_2(x) = \frac{1}{2}$, $x \in [\frac{1}{3}, \frac{1}{2}]$.

Then ϕ_2 is continuous on $[\frac{1}{3}, \frac{1}{2}]$, f is bounded on $[\frac{1}{3}, \frac{1}{2}]$

and $f = \phi_2$ except at $\frac{1}{3}$

$$\therefore \int_{1/3}^{1/2} f = \int_{1/3}^{1/2} \phi_2 = \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{3})$$

... ...

similarly, $\int_{\frac{1}{n+1}}^{\frac{1}{n}} f = \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

Now

$$\int_0^1 f = \lim_{n\to\infty}\left[1(1-\tfrac{1}{2}) + \tfrac{1}{2}(\tfrac{1}{2}-\tfrac{1}{3}) + \tfrac{1}{3}(\tfrac{1}{3}-\tfrac{1}{4}) + \cdots + \tfrac{1}{n}\left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) - \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right] - \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)$$

$= \frac{\pi^2}{6} - 1$ Ans., since $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

References:- Introduction to Real Analysis - S.K. Mapa.

- Fundamenals of Mathematical Analysis - I. Hoque.