E- LEARNING MATERIALS FOR SEM - 6(I)

UNIT - 3 , CC - 14

TOPIC - RING THEORY AND LINEAR ALGEBRA - II

DATE - 24.04.2020

PREPAIRED BY - DR. ALAUDDIN DAFADAR

Inner product space or Euclidean space :⇒ P-184

Real inner product :⇒ Let V be a real vector space. A real inner product on V is a mapping $f: V \times V \longrightarrow \mathbb{R}$ that assigns to each ordered pair of vectors (α, β) of V a real no. $f(\alpha, \beta)$, generally denoted by $\alpha \cdot \beta$ or by (α, β), satisfying the following properties ⟶

1) $(\alpha, \beta) = (\beta, \alpha)$ for all $\alpha, \beta \in V$, (symmetry)
2) $(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma)$ for all $\alpha, \beta, \gamma \in V$ (linearity)
3) $(c\alpha, \beta) = c(\alpha, \beta) = (\alpha, c\beta)$ for all $\alpha, \beta \in V$ and all $c \in \mathbb{R}$ (homogeneity)
4) $(\alpha, \alpha) > 0$ if $\alpha \neq \theta$ (Positivity)

If $\alpha = \theta$ then $(\alpha, \alpha) = 0$.

The mapping f is also denoted by $(,)$

A real vector space V together with a real inner product defined on it, is said to be a Euclidean space.

A Euclidean space is also called an inner product space.

Ex :⇒ In $\mathbb{R}^2$, the standard inner product is defined by $(\alpha, \beta) = a_1 b_1 + a_2 b_2$ where $\alpha = (a_1, a_2) \in \mathbb{R}^2$, $\beta = (b_1, b_2) \in \mathbb{R}^2$

In $\mathbb{R}^2$ let us define $(\alpha, \beta) = 2a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$

Then $(,)$ satisfies the conditions 1, 2, 3 of a real inner product.

$$(\alpha, \alpha) = 2a_1^2 + 2a_1a_2 + a_2^2$$
$$= a_1^2 + (a_1 + a_2)^2 > 0 \text{ except } a_1 = a_2 = 0$$

$\therefore (\alpha, \alpha) > 0$ for $\alpha \neq \theta$.

Thus $(\alpha,\beta) = 2a_1 + a_1b_2 + a_2b_1 + a_2b_2$ defines a real inner product in $\mathbb{R}^2$.

2

Norm of a vector :⇒

If α be a vector in a Euclidean space V with the inner product $(\cdot,\cdot)$, the norm of α, denoted by $\|\alpha\|$, is defined by
$\|\alpha\| = \sqrt{(\alpha,\alpha)}$

** Let α be a vector in Euclidean space V and $\|\alpha\|$ be its norm.

Then 1) $\|c\alpha\| = |c|\|\alpha\|$, c being a real no.

2) $\|\alpha\| > 0$ unless $\alpha = \theta$, $\|\theta\| = 0$

Proof :⇒ i) $\|c\alpha\| = \sqrt{(c\alpha, c\alpha)} = \sqrt{c^2(\alpha,\alpha)} = |c|\sqrt{(\alpha,\alpha)}$
$= |c|\,\|\alpha\|.$

ii) $\alpha \neq \theta$ implies $(\alpha,\alpha) > 0$ and therefore $\|\alpha\| > 0$

If $\alpha = \theta$ then $(\alpha,\alpha) = (\theta,\theta) = 0$, therefore $\|\alpha\| = 0$

Schwarz's inequality :⇒ F - 186

For any two vectors α, β in a Euclidean space V,
$|(\alpha,\beta)| \leq \|\alpha\|\|\beta\|$,
the equality holds when α, β are linearly dependent

⇒ Case - I Let one or both of α, β be null.
Then both sides being zero, the equality holds

Case - II Let α, β non-null and linearly dependent.
Then there exists a non-zero real number K such that $\alpha = K\beta$

Then $\|\alpha\| = |K|\|\beta\|$ and $(\alpha,\beta) = (K\beta,\beta) = K\|\beta\|^2$

Therefore $|(\alpha,\beta)| = |K|\|\beta\|^2 = \|\alpha\|\|\beta\|$

<u>Case - III</u> : Let α, β be not linearly dependent. Then $\alpha - K\beta \neq \theta$ for all real K.

$\therefore$ $(\alpha - K\beta, \alpha - K\beta) > 0$ for all real K

or, $(\alpha, \alpha) - (\alpha, K\beta) - (K\beta, \alpha) + K^2(\beta, \beta) > 0$ for all real K

or, $(\alpha, \alpha) - 2K(\alpha, \beta) + K^2(\beta, \beta) > 0$, for all real K

Since (α, α), (α, β), (β, β) are all real and $(\beta, \beta) \neq 0$, the left hand side is a real quadratic polynomial in K, and since it is positive for all real values of K,

The discriminant of the quadratic polynomial must be negative, for otherwise the polynomial would be zero for some real K.

Thus $(\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta) < 0$, where $|(\alpha, \beta)| < \|\alpha\| \|\beta\|$

This complet the proof.

<u>Theorem 01</u> :⇒ In a Euclidean space V, two vectors α, β are linearly dependent if and only if $|(\alpha, \beta)| = \|\alpha\| \|\beta\|$

⇒ Let α, β be linearly dependent.

If one or both α, β be null, then the equality holds.

If α, β both non-nul, ~~then the equality holds~~.

Then $\alpha = K\beta$ for some non-zero real K.

In this case $\|\alpha\| = |K| \|\beta\|$ and $(\alpha, \beta) = (K\beta, \beta) = K(\beta, \beta) = K\|\beta\|^2$

$\therefore$ $|(\alpha, \beta)| = |K| \|\beta\|^2 = \|\alpha\| \|\beta\|$

<u>Conversely</u> :⇒ let $|(\alpha, \beta)| = \|\alpha\| \|\beta\|$

α, β are linearly independent implies $|(\alpha, \beta)| < \|\alpha\| \|\beta\|$ by 'schwarz's inequality'

Contrapositively, $|(\alpha, \beta)| \nless \|\alpha\| \|\beta\|$ implies α, β are linearly dependent.

But by Schwarz's inequality, $|(\alpha,\beta)| \leq ||\alpha|| \, ||\beta||$ for all α, β in V. 1

$\therefore \; |(\alpha,\beta)| = ||\alpha|| \, ||\beta||$

$\Rightarrow \alpha, \beta$ are linearly independent.

Orthogonal :⇒ A set of vectors $\{\beta_1, \beta_2, \ldots, \beta_r\}$ in a Euclidean space is said to be orthogonal if $(\beta_i, \beta_j) = 0$ where $i \neq j$

Orthonormal :⇒ A set of vectors $\{\beta_1, \beta_2, \ldots, \beta_r\}$ in a Euclidean space is said to be orthonormal if

$$(\beta_i, \beta_j) = 0 \text{ for } i \neq j$$
$$= 1 \text{ for } i = j$$

* An orthogonal set of vectors may contain the null vector θ but an orthonormal set contains only non-null vectors.

P-188 S K Mapa

Theorem 01 : An orthogonal set of non-null vectors in a Euclidean space V is linearly independent.

⇒ Let $\{\beta_1, \beta_2, \ldots, \beta_r\}$ be an orthogonal set of non-null vectors.

Let us consider the relation

$c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r = \theta$, where c_i are real no.

Then $(c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r, \beta_i) = (\theta, \beta_i) = 0$ for $i = 1, 2, \ldots, r$.

or, $c_1(\beta_1, \beta_i) + c_2(\beta_2, \beta_i) + \cdots + c_r(\beta_r, \beta_i) = 0$

or $c_i(\beta_i, \beta_i) = 0$ since $(\beta_i, \beta_j) = 0$, $j \neq i$

Since β_i is non-null, $(\beta_i, \beta_i) > 0$ and therefore $c_i = 0$.

This prove that the set $\{\beta_1, \beta_2, \ldots, \beta_r\}$ is linearly independent.

Scalar component and projection :⇒

Let β be a fixed non-zero vector in a Euclidean space V. Then for a non-zero vector α in V there exists a unique real no. c such that $\alpha - c\beta$ is orthogonal to β.

c is denoted by $(\alpha - c\beta, \beta) = 0$. Therefore $(\alpha, \beta) = c(\beta, \beta)$.

giving $c = \dfrac{(\alpha, \beta)}{(\beta \cdot \beta)}$.

In here c is said to be scalar component of α along β and $c\beta$ is said to be projection of α upon β.

Bessel's inequality :⇒ P-191

If $\{\beta_1, \beta_2, \ldots, \beta_r\}$ be an ~~orthogonal~~ orthonormal set of vectors in a Euclidean space V, then for any vector α in V.

$$\|\alpha\|^2 \geq c_1^2 + c_2^2 + \ldots + c_r^2$$

where c_i is the scalar component of α along β_i. $i = 1, 2, \ldots, r$

⇒ For all $i \ (i = 1, 2, \ldots, r)$, $c_i = (\alpha, \beta_i)$, since $(\beta_i, \beta_i) = 1$

$\alpha - c_1\beta_1 - c_2\beta_2 - \ldots - c_r\beta_r$ is orthogonal to each β_i $1 \leq i \leq r$, since

$$(\alpha - c_1\beta_1 - c_2\beta_2 - \ldots - c_r\beta_r, \beta_i) = (\alpha, \beta_i) - c_i(\beta_i, \beta_i) = 0$$

It follows that $\alpha - c_1\beta_1 - c_2\beta_2 - \ldots - c_r\beta_r$ is orthogonal to $c_1\beta_1 + c_2\beta_2 + \ldots + c_r\beta_r$

$$\alpha = (\alpha - c_1\beta_1 - c_2\beta_2 - \ldots - c_r\beta_r) + (c_1\beta_1 + c_2\beta_2 + \ldots + c_r\beta_r)$$

Scanned with CamScanner

By pythagoras theorem

$\|\alpha\|^2 = \|\alpha - c_1\beta_1 - c_2\beta_2 - \cdots - c_r\beta_r\|^2 + \|c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r\|^2$

$\geq \|c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r\|^2$, since a norm is non-negative.

But $\|c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r\|^2$

$= (c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r, c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r)$

$= c_1^2 + c_2^2 + \cdots + c_r^2$ [since $\{\beta_1, \beta_2, \ldots, \beta_r\}$ is an ~~orthogonal~~ orthonormal set therefore $(\beta_i, \beta_j) = 0, \; i \neq j$]
$(\beta_i, \beta_j) = 1, \; i = j$]

Consequently, $\|\alpha\|^2 \geq c_1^2 + c_2^2 + \cdots + c_r^2$.
This complet the proof.

* The equality occurs if $\|\alpha - c_1\beta_1 - c_2\beta_2 - \cdots - c_r\beta_r\|^2 = 0$
i.e.; if $\alpha = c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r$, i.e.; if
$\alpha \in L\{\beta_1, \beta_2, \ldots, \beta_r\}$.

Theorem 02 : Parseval's Theorem P-191

If $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be an orthonormal basis of a Euclidean space V, then for any vector α in V,

$$\|\alpha\|^2 = c_1^2 + c_2^2 + \cdots + c_n^2$$

which c_i is the scaler component of α along β_i, $i = 1, 2, \ldots, n$.

$\Rightarrow$ Since $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a basis of V, any vector $\alpha \in V$ can be expressed as $\alpha = c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n$ where c_i is the scaler component of α along β_i, $i = 1, 2, \ldots, n$.

Scanned with CamScanner

So $\|\alpha\|^2 = (\alpha, \alpha) = (c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n, c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n)$

$= c_1^2 + c_2^2 + \cdots + c_n^2$, since $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is an orthonormal set.

This completes the proof.

Ex. 1 Extend the set of vectors $\{(2,3,-1), (1,-2,-4)\}$ to an orthogonal basis of the Euclidean space $\mathbb{R}^3$ with standard inner product and then find the associated orthonormal basis.

$\Rightarrow$ Let $\alpha_1 = (2,3,-1)$, $\alpha_2 = (1,-2,-4)$

α_1 and α_2 orthogonal vectors. Let $\alpha_3 = (0,0,1)$.

Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent because

$$\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & -4 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

So $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $\mathbb{R}^3$.

Let $\beta = \alpha_3 - c_1\alpha_1 - c_2\alpha_2$, where c_1 and c_2 are real no.

$$c_1 = \frac{(\alpha_3, \alpha_1)}{(\alpha_1, \alpha_1)}, \quad c_2 = \frac{(\alpha_3, \alpha_2)}{(\alpha_2, \alpha_2)}$$

Then β is orthogonal to α_1 and α_2 and

$L\{\alpha_1, \alpha_2, \alpha_3\} = L\{\alpha_1, \alpha_2, \beta\}$

$\therefore \{\alpha_1, \alpha_2, \beta\}$ is an orthogonal basis of $\mathbb{R}^3$.

$$c_1 = \frac{((0,0,1), (2,3,-1))}{((2,3,-1), (2,3,-1))} = -\frac{1}{14}$$

$$c_2 = \frac{((0,0,1), (1,-2,-4))}{((1,-2,-4), (1,-2,-4))} = -\frac{4}{21}$$

$\therefore \beta = (0,0,1) + \frac{1}{14}(2,3,-1) + \frac{4}{21}(1,-2,-4) = (\frac{1}{3}, -\frac{1}{6}, \frac{1}{6})$

Hence an extended orthogonal basis is
$\{(2,3,-1), (1,-2,-4), (\frac{1}{3}, -\frac{1}{6}, \frac{1}{6})\}$ and associated orthonormal basis is

$$\left\{\frac{1}{\sqrt{14}}(2,3,-1), \frac{1}{\sqrt{21}}(1,-2,-4), \cancel{(\frac{1}{3}, -\frac{1}{6}, \frac{1}{6})}\ \frac{1}{\sqrt{6}}(2,-1,1)\right\}$$

Theorem 03:

Every non-null subspace W of a finit dimensional Euclidean space V possesses an orthogonal basis.
(Gram-schmidt process).

⇒ Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a basis of W.
Since the basis vectors are non-zero,
we pick up one of them, say α_1, and consider as the first member of the new basis.

Let $\beta_1 = \alpha_1$

Let $\beta_2 = \alpha_2 - c_1\beta_1$, where $c_1\beta_1$ projection of α_2 upon β_1.

Then β_2 is orthogonal to β_1 and
$L\{\beta_1, \beta_2\} = L\{\beta_1, \alpha_2\} = L\{\alpha_1, \alpha_2\}$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \cdot \beta_1$$

$\alpha_3 \notin L\{\beta_1, \beta_2\}$. Let $\beta_3 = \alpha_3 - d_1\beta_1 - d_2\beta_2$, where $d_1\beta_1, d_2\beta_2$ are the projections of α_3 upon β_1, β_2 respectively.

Then β_3 is orthogonal to β_1, β_2 and
$L\{\beta_1, \beta_2, \beta_3\} = L\{\beta_1, \beta_2, \alpha_3\} = L\{\alpha_1, \alpha_2, \alpha_3\}$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2$$

$\alpha_4 \notin L\{\beta_1, \beta_2, \beta_3\}$. Let $\beta_4 = \alpha_4 - r_1\beta_1 - r_2\beta_2 - r_3\beta_3$ where

$r_1\beta_1, r_2\beta_2, r_3\beta_3$ are projections of α_4 upon $\beta_1, \beta_2, \beta_3$ respectively

Then β_4 is orthogonal to $\beta_1, \beta_2, \beta_3$ and

~~$L\{\beta_4, \beta_3, \beta_2, \beta_1\} = L\{\alpha_4, \alpha_3, \alpha_2, \alpha_1\}$~~

$L\{\beta_1, \beta_2, \beta_3, \beta_4\} = L\{\beta_1, \beta_2, \beta_3, \alpha_4\} = L\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

$$\beta_4 = \alpha_4 - \frac{(\alpha_4, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_4, \beta_2)}{(\beta_2, \beta_2)}\beta_2 - \frac{(\alpha_4, \beta_3)}{(\beta_3, \beta_3)}\beta_3$$

This process terminates after a finit number of steps because at every step one vector of the given basis is replaced by a vector in the desired orthogonal basis. Finally we obtained

$$\beta_r = \alpha_r - \frac{(\alpha_r, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_r, \beta_2)}{(\beta_2, \beta_2)}\beta_2 - \cdots - \frac{(\alpha_r, \beta_{r-1})}{(\beta_{r-1}, \beta_{r-1})}\beta_{r-1}$$

and $\{\beta_1, \beta_2, \ldots, \beta_r\}$ is an orthogonal basis of the supspace W.

This complet the proof.

REFERENCES - HIGHER ALGEBRA - 8

S.K. MAPA.