Sequence Of Function

Definition : Let $D \subseteq R$ and for each $n \in N$, let $f_n : D \to R$ be a function, then $\{f_n\}$ is said to be sequence of function on D to R. D is said to be the domain of the sequence of functions $\{f_n\}$. The terms of the sequence of function are $\{f_1(x),\ f_2(x),\ f_3(x).......,f_n(x),............\}$.

Note :

(*i*) Sequence of function is a function for which each term of the sequence is a function itself.

(*ii*) The domain $D \subseteq R$, which is not only natural numbers (in case real sequence), which may be a close interval $[a,b]$ or open interval (a,b)

(*iii*) For each $x_0 \in D$, the sequence $\{f_n\}$ gives rise to a sequence of real numbers $\{f_n(x_0)\}$, which is obtained by evaluating each f_n at x_0.

(*iv*) Converges and diverges (diverging) of the sequence of function $\{f_n(x)\}$ depending only on $x \in D$.

Thus, for some $x \in D$ this may be converges and for other $x \in D$ this may be diverges.

Example: Let for each $n \in N$, $f_n : R \to R$ be a sequence of function.

Where $f_n(x) = x^n$, $x \in R$.

Since, $\{f_n(x)\}$ converges to 0 if $-1 < x < 1$.

And $\{f_n(x)\}$ converges to 1 if $x = 1$

So, for all other $x \in R$, the sequence $\{f_n(x)\}$ is divergent.

Point wise Convergence : Let $D \subseteq R$ and for each $n \in N$, let $f_n : D \to R$ be a function. Then the sequence of function $\{f_n\}$ is said to be point wise converges on D if for each $x \in D$, the sequence $\{f_n(x)\}$ converges.

Otherwise

Let $D \subseteq R$ and for each $n \in N$, let $f_n : D \to R$ be a function. Then the sequence of function $\{f_n\}$ is said to be point wise convergent on D to f, if for given $\in > 0$, ∃ a natural number k (*depending on* $\in$ *and* x *both*) such that

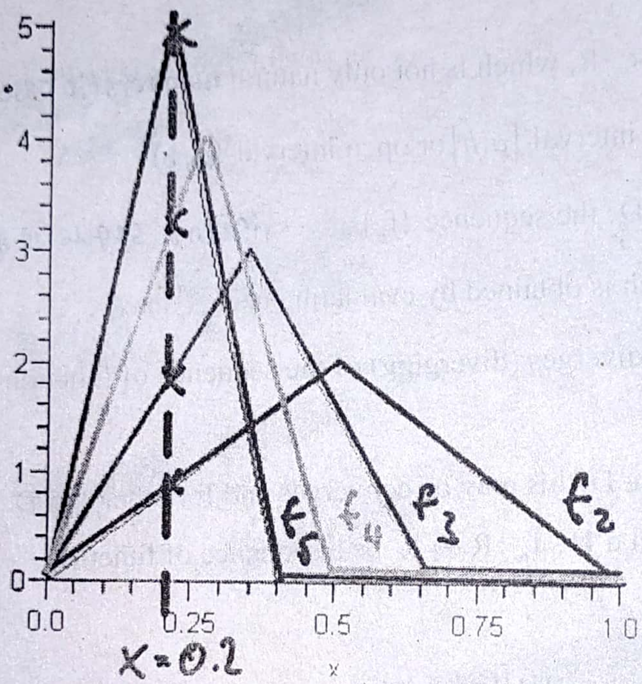
$$\forall x \in [a,b],\ |f_n(x) - f(x)| < \in,\quad \forall n \geq k.$$

Let the sequence $\{f_n\}$ be point wise convergent on D and let $c \in D$. Then the sequence $\{f_n(c)\}$ is convergent. Let $\lim_{n\to\infty} f_n(c) = l_c$. Since for all $x \in D$, $\{f_n(x)\}$ converges to a limit, l_x exists for all $x \in D$.

Let us define a function $f : D \to R$ by $f(x) = l_x$, $x \in D$. Then f is said to be the limit function of the sequence $\{f_n\}$ on D.

In this case we also say that the sequence $\{f_n\}$ converges to f on D and we write $f = \lim_{n\to\infty} f_n$ on D, or $f_n \to f$ on D.

Example: Define $\{ f_n(x) \}$, where $f_n(x) = max(n - n^2 |x - 1/n|, 0)$ and $x \in [0, 1]$. Describe the elements of this function sequence.



If we fix a value for x, say $x = 0.2$, we get a numeric sequence. In the picture above you can see that

$x_2 = f_2(0.2) = 0.8$, $x_3 = f_3(0.2) = 1.8$, $x_4 = f_4(0.2) = 3.2$, and $x_5 = f_5(0.2) = 5.0$

At first glance looks like $x_n = f_n(0.2)$ is a sequence diverging to infinity, but looking at the pattern for the functions f_n we guess that x_n must eventually be zero for $n > N$. Thus the sequence $\{ x_n \}$ should converge to zero (this is of course the sequence from the above example).

Uniform Convergence : Let $D \subseteq R$ and for each $n \in N$, let $f_n : D \to R$ be a function. Then the sequence $\{f_n\}$ is said to be uniformly convergent on D to a function f if for given $\in > 0$ $\exists$ a positive integer m (depending on $\in$ only) such that

$$\forall x \in D, |f_n(x) - f(x)| < \in, \quad \forall n \geq m.$$

i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on D.

i.e. $f_n \to f$ uniformly on D.

This case, f is said to be the uniform limit of the sequence $\{f_n\}$ on D.

Note-1 :Every uniformly convergent sequence is point-wise convergent and the uniform limit function is same as pointwise limit function, but not conversely.

Note-2 : Non-point wise convergence $\Rightarrow$ non-uniformly convergence.

i.e. a sequence which is not pointwise convergence can not be uniformly convergent.

Ex-: Let $\{f_n(x)\}$ be a sequence of function.

, where $f_n(x) = x^n$, $x \in R$.

Then, $f(x) = 0, -1 < x < 1.$

$= 1, x = 1$

Here we see that

(*i*) The domain of the sequence $\{f_n\}$ is R, but the domain of pointwise convergence of the sequence is a proper subset of R.

(*ii*) The function $f_n(x)$ is continuous on R but the limit function f(x) is not continuous there.

Ex- : Let $D = \{x \in R : x \geq 0\}$ and for each $n \in N$, let $f_n : [a,b] \to R$ be defined by

$$f_n(x) = \frac{n}{n+x}, \quad x \geq 0$$

$$\therefore \lim_{n\to\infty} f_n(x) = f(x) = \lim_{n\to\infty} \frac{n}{n+x} = 1$$

$\therefore$ The sequence is point wise convergent on D to the function f (limit function) defined by

$$f(x) = 1, \ \forall x \geq 0.$$

Ex- : Let $\{f_n(x)\}$ be a sequence of function.

, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in R$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\ldots\}$, this converges to 0.

$$\text{For } x \neq 0, \ \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} = 0.$$

$\therefore$ The sequence $\{f_n\}$ is point wise convergent on R to the function f defined by

$f(x) = 0, \ x \in R$.

Ex-: Let $D = \{x \in R : x \geq 0\}$ and for each $n \in N$,

let $f_n : D \to R$ be defined by $f_n(x) = \frac{nx}{1+nx}$.

Then, $\{f_n(x)\}$ is a sequence of function on D.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\ldots\ldots\}$, this converges to 0.

For $x > 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x} = 1$.

$\therefore$ The sequence is point wise convergent on D to the function f defined by

$$f(x) = 0, \quad x = 0.$$
$$= 1, \quad x > 0.$$

Ex: Let $\{f_n\}$ be a sequence of function.

, where $f_n(x) = \tan^{-1} nx, x \in R$

$$\text{Then, } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} nx = \begin{cases} \pi/2 & , x > 0 \\ 0 & , x = 0 \\ -\pi/2 & , x < 0 \end{cases}$$

$\therefore$ The sequence $\{f_n\}$ is point-wise convergent on R to the function f.

,where $f(x) = \frac{\pi}{2} \operatorname{sgn} x, \quad x \in R$.

Ex: Show that every point wise convergent sequence is not uniformly convergent.

Solution : We prove this by taking an example.

Let $\{f_n\}$ be a sequence of function. ,where $f_n = \frac{nx}{1+n^2x^2}, x \in R$

$$\text{Here, } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x/n}{\frac{1}{n^2}+x^2} = 0, \ \forall x \in R$$

$\therefore$ The sequence $\{f_n\}$ is point wise convergent with point wise limit f such that

$$f(x) = 0, \ \forall x \in R$$

we shall now show that the convergence is not uniform in any interval $[a,b]$ with 0 as an interior point.

Let us suppose that $\{f_n\}$ is uniformly convergent in $[a,b]$ so that the point wise limit f is also the uniform limit.

Then by definition, for $\epsilon > 0$, $\exists$ a positive integer m (depending on ϵ alone) such that

$\forall x \in [a,b]$, $\left|\frac{nx}{1+n^2x^2} - 0\right| < \epsilon, \quad \forall n \geq m$

we take, $\varepsilon = \frac{1}{4}$. Now $\exists$ an integer k such that $k \geq m$ and $\frac{1}{k} \in [a,b]$

Taking $n = k$ and $x = \frac{1}{k}$, we have

$$\frac{nx}{1+n^2x^2} = \frac{1}{2} < \frac{1}{4} = \epsilon$$

Thus, we arrive a contradiction and this contradiction shows that $\{f_n\}$ is not uniformly convergent in any interval $[a,b]$ with 0 as an interior point even though it is point wise convergent there.

Ex: For each $n \in \mathbb{N}$, let $f_n : (-1,1] \to R$ be defined by $f_n(x) = x^n, x \in (-1,1]$. Then the sequence $\{f_n\}$ is point-wise convergent on $(-1,1]$ to the function f where

$f(x) = 0$ for $-1 < x < 1$

$= 1$ for $x = 1$

Let us examine if the convergence of the sequence $\{f_n\}$ is uniform on $(0,1)$.

Let $c \in (0,1)$. Then $|f_n(c) - f(c)| = c^n$.

Let $0 < \varepsilon < 1$. Then $|f_n(c) - f(c)| < \varepsilon$ whenever $c^n < \varepsilon$,

i.e., whenever $n \log\left(\frac{1}{c}\right) > \log\left(\frac{1}{\varepsilon}\right)$,

i.e., whenever $n > \dfrac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{c}\right)}$

Let $k = \left[\dfrac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{c}\right)}\right] + 1$. Then k is a natural number and $|f_n(c) - f(c)| < \varepsilon$ for all $n \geq k$.

Therefore, for all $x \in (0,1), |f_n(x) - f(x)| < \varepsilon$ for all $n \geq k$ where $k = \left[\dfrac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{x}\right)}\right] + 1$.

Thus k depends on ε as well as on x. As $x \to 1^-, k \to \infty$.

It follows that does not exist a natural number k such that for all $x \in (0,1), |f_n(x) - f(x)| < \varepsilon$ holds for all $n \geq k$. Consequently, $\{f_n\}$ is not uniformly convergent on $(0,1)$

Let $a \in R$ such that $0 < a < 1$.

In $[0,a]$, the greatest value of $\left[\frac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{x}\right)}\right]$ is $\left[\frac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{a}\right)}\right]$.

Let $k=\left[\frac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{1}{a}\right)}\right]+1$. Then k is a natural number and for all $x\in[0,a]$.

Then for all $|f_n(x)-f(x)|<\varepsilon$ for all $n\geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0,a]$.

Cauchy's Criterion for Uniformly Convergence :

Statement: A sequence of function $\{f_n\}$ defined on [a,b] converges uniformly on [a,b] if and only if for every $\in>0$ and for all $x\in$ [a,b], $\exists$ a positive integer m such that

$$|f_{n+p}(x)-f_n(x)|<\in,\quad \forall n\geq m,\ p=1,2,3,\ldots\ldots$$ V.H.', 2010, 2012

Proof: Necessary Part

Let the sequence $\{f_n\}$ uniformly converges on [a,b] to the limit function f, so that for a given $\in>0$ and for all $x\in$ [a,b], $\exists$ positive integers m_1, m_2 such that

$$|f_n(x)-f(x)|<\frac{\in}{2},\quad \forall n\geq k \quad\ldots\ldots\ldots\ldots (1)$$

Thus, for all $x\in[a,b]$ $|f_{n+p}(x)-f(x)|<\frac{\in}{2},\ \forall n\geq k, p=1,2,3,\ldots\ldots\ldots (2)$

Then, $|f_{n+p}(x)-f_n(x)| = |f_{n+p}(x)-f(x)+f(x)-f_n(x)|$

$$\leq |f_{n+p}(x)-f(x)| + |f_n(x)-f(x)|$$

$$<\frac{\in}{2}+\frac{\in}{2},\quad \text{by (1) and (2).}$$

$$=\in$$

$\therefore |f_{n+p}(x)-f_n(x)|<\in,\ \forall n\geq k,\ p=1,2,3,\ldots\ldots$

Sufficient Part

Let the given condition is hold i.e. for given $\in>0$, $\exists$ a positive integer m such that

$\forall x\in [a,b]$, $|f_{n+p}(x)-f_n(x)|<\in,\ \forall n\geq k, p=1,2,3,\ldots\ldots$

Let $x_0\in [a,b]$ be a point.

Then, $|f_{n+p}(x_0)-f_n(x_0)|<\in,\ \forall n\geq m, p=1,2,3,\ldots\ldots$

It follows that the sequence $\{f_n(x_0)\}$ is a Cauchy's sequence in R and so it is convergent.

Consequently, the sequence $\{f_n\}$ is point wise convergent on [a,b].

Let the limit function be f.

Now, we have,

$$\forall x \in [a,b],\ |f_{n+p}(x) - f_n(x)| < \epsilon/2,\ \ \forall n \geq k,\ p = 1,2,3,\ldots\ldots$$

$$\therefore \forall x \in [a,b],\ |f_{k+p}(x) - f_k(x)| < \epsilon/2,\ \ p = 1,2,3,\ldots\ldots$$

i.e. $f_k(x) - \epsilon/2 < f_{k+p}(x) < f_k(x) + \epsilon/2 \quad for\ p = 1,2,3,\ldots\ldots$

.... (3)

Since, $\lim_{p\to\infty} f_{m+p}(x) = f(x)$.

Now, taking limit as $p \to \infty$ in (3), we have $\forall x \in [a,b]$

$$f_m(x) - \epsilon/2 \leq f(x) \leq f_m(x) + \epsilon/2,\ \ \forall x \in [a,b]$$

$$Or,\ |f_m(x) - f(x)| \leq \epsilon/2 < \epsilon,\ \ \forall x \in [a,b]$$

Similarly, the inequality holds for $k+1, k+2, \ldots\ldots\ldots\ldots$

$\therefore$ For all $x \in [a,b]$, $|f_n(x) - f(x)| < \epsilon,\ \forall n \geq k.$

This proves that the sequence $\{f_n\}$ is uniformly convergent on [a,b].

Hence the theorem.

Cauchy's criterion for uniform convergence (Another form):

Theorem: Let $[a,b]$ be a closed and bounded interval and let $\{f_n\}$ be a sequence of functions on $[a,b]$ to R. A necessary and sufficient condition for uniform convergence of the sequence $\{f_n\}$ on $[a,b]$ is that for a pre-assigned positive ε there exists a natural number k (depending only on ε) such that for all $x \in [a,b]$, $|f_m(x) - f_n(x)| < \varepsilon$ for all $m,n \geq k$. C.H.2009

Proof: The condition is necessary: Let the sequence $\{f_n\}, n \in N$ be uniformly convergent on $[a,b]$ to R and let the limit function be f. Then for a pre- assigned positive ε there exists a natural number k (depending on ε) such that for all $x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $n \geq k$.

Thus if $m,n \geq k$, we have for all $x \in [a,b]$,

$$|f_m(x) - f_n(x)| = |\{f_m(x) - f(x)\} - \{f_n(x) - f(x)\}|$$

$$\leq |f_m(x)-f(x)|+|f_n(x)-f(x)|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

The condition is sufficient: Conversely, let the condition be satisfied. Then for a chosen $\varepsilon>0$ there exists a natural number k such that for all $x\in[a,b]$, $|f_m(x)-f_n(x)|<\varepsilon$ for all $m,n\geq k$.

Let $x_0\in[a,b]$. Then $|f_m(x_0)-f_n(x_0)|<\varepsilon$ for all $m,n\geq k$.

It follows that the sequence $\{f_n(x_0)\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore it is convergent. Consequently, the sequence is point-wise convergent on $[a,b]$. Let the limit function be f.

Let us choose $\varepsilon>0$. Then by the condition, there exists a natural number k such that for all $x\in[a,b]$, $|f_m(x)-f_n(x)|<\frac{\varepsilon}{2}$ for all $m,n\geq k$

Keeping n fixed and taking $m\to\infty$ we have, for all $x\in[a,b]$, $|f(x)-f_n(x)|\leq\frac{\varepsilon}{2}<\varepsilon$

Thus for each $\varepsilon>0$ there exists a natural number k such that for all $x\in[a,b]$, $|f_n(x)-f(x)|<\varepsilon$ for all $n\geq k$

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$

Theorem : V.H.',96, 2k.2011, C.H. '04,'06

Statement: Let $D\subseteq R$ and for each $n\in N$, let $f_n:D\to R$ is continuous on D. If the sequence $\{f_n\}$ is uniformly convergent on D to a function f, then f is continuous on D.

Proof : Let $c\in D$. Let us choose $\in>0$.

Since $\{f_n\}$ is uniformly convergent on D to the function f, $\exists$ a natural number k (depending on $\in$ only) such that

$$\forall x\in D, |f_n(x)-f(x)|<\in/3 \quad ,\forall n\geq k$$

$$\therefore |f_k(x)-f(x)|<\in/3 \quad ,\forall x\in D \ \ldots\ldots (1)$$

Since, $c\in D$ $\therefore |f_k(c)-f(c)|<\in/3$ .... (2)

Also, since each f_n be continuous on D.

$\therefore$ f_k is continuous at $c\in D$, $\exists$ a positive δ such that

$$|f_k(x)-f_k(c)|<\in/3 \quad , \text{ whenever } |x-c|<\delta \ \ldots\ldots (3)$$

Now, $|f(x)-f(c)| = |f(x)-f_k(x)+f_k(x)-f_k(c)+f_k(c)-f(c)|$

$$\leq |f_k(x)-f(x)|+|f_k(x)-f_k(c)|+|f_k(C)-f(c)|$$