Semester-IV

Course Type - core-8

Course Title - C8T sequence of function.

Topic - Sequence of function

References - S. K. Mapa book.

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— Shambhu Nath Acharya

$$f(x)=\begin{cases}0, & x=0\\ \frac{1}{x}, & 0<x\le 1.\end{cases}$$

Hence the function $\{f_n\}_n$ is point wise convergent on $[0,1]$, but the convergent is not uniform on $[0,1]$ as the limit function f is not continuous on $[0,1]$. (Proved)

A Test for Uniform Convergence of Sequence of function :

Theorem : (M_n – Test) [V.H.10]

Statement : Let $\{f_n\}$ be a sequence of function defined on [a,b] such that $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in [a,b]$ and let $M_n = \text{Sup}\{|f_n(x) - f(x)| : x \in [a,b]\}$

Then, $\{f_n\}$ converges uniformly on [a,b] if and only if $M_n \to 0$ as $n \to \infty$.

- **Proof :** Necessary Part

Let $\{f_n\}$ be uniformly converges to f on [a,b].

Then for a given $\in > 0$, $\exists$ a positive integer m such that

$$|f_n(x) - f(x)| < \in, \quad \forall n \ge m \text{ and } \forall x \in [a,b].$$

$$\Rightarrow \text{Sup}\{|f_n(x) - f(x)| : x \in [a,b]\} < \in, \quad \forall n \ge m.$$

$$\Rightarrow M_n < \in, \quad \forall n \ge m$$

$$\Rightarrow |M_n - 0| < \in, \quad \forall n \ge m$$

$$\Rightarrow M_n \to 0 \text{ as } n \to \infty$$

Sufficient Part : Let $M_n \to 0$ as $n \to \infty$.

Then for a given $\in > 0$, $\exists$ a positive integer m such that

$$|M_n - 0| < \in, \quad \forall n \ge m$$

$$\Rightarrow M_n < \in, \quad \forall n \ge m$$

$$\Rightarrow \text{Sup}\{|f_n(x) - f(x)| : x \in [a,b]\} < \in, \quad \forall n \ge m.$$

$$\Rightarrow |f_n(x) - f(x)| < \in, \quad \forall n \ge m \text{ and } \forall x \in [a,b]$$

$\Rightarrow$ $\{f_n\}$ is uniformly convergent to f on [a,b].

Ex : Let $f_n(x) = x^n$, $x \in [0,1]$.

Show that the sequence of function $\{f_n\}$ is not uniformly convergent on [0,1], but is uniformly convergent on [0,a] where $0 < a < 1$.

Solution : First Part

Since for $0 \le x < 1$, $\lim_{x\to\infty} f_n(x) = 0$.

For $x = 1$, the sequence is $\{1, 1, \ldots\ldots.\}$, this converges to 1.

$\therefore$ The sequence $\{f_n\}$ converges to the function f.

where, $f(x) = 0,\ 0 \le x < 1$

$= 1,\ x = 1$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$

$= 1$

$\therefore \lim_{n\to\infty} M_n = 1 \ne 0$.

By M_n – Test the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Second part:

In this case, $f_n(x) = x^n,\ x \in [0,a],\ 0 < a < 1$.

$\because \lim_{n\to\infty} f_n(x) = 0,\ \forall x \in [0,a]$.

Thus the limit of function is

$$f(x) = 0,\ \forall x \in [0,a]$$

Let $M'_n = \sup_{x\in[0,a]} |f_n(x) - f(x)|$

$= \sup_{x\in[0,a]} x^n = a^n,\ \forall n \in N$.

$\therefore \lim_{n\to\infty} M'_n = \lim_{n\to\infty} a^n = 0$, as $0 < a < 1$.

This shows that the sequence of function $\{f_n\}$ uniformly convergent on $[0,a]$, $0 < a < 1$.

Ex: Show that the sequence $\{f_n\}$ of functions where $f_n(x) = \dfrac{n}{n+x}$, is uniformly convergent in $[0,k]$, whatever k may be, but not uniformly convergent in $[0,\infty)$. **[C.H.'04]**

Solution: Since the sequence of function $\{f_n\}$ is point-wise convergent $\forall x \ge 0$ and the point-wise limit f is given by $f(x) = 1,\ \forall x \ge 0$

Let $\in > 0$ be given., we have

$$|f_n(x) - f(x)| = \frac{x}{n+x} < \in\ ,\quad \text{if } n > x\left(\frac{1}{\in} - 1\right)$$

Let $m(\in, x)$ denote the positive integer just greater than $x\left(\dfrac{1}{\in} - 1\right)$.

Obviously, $m(\varepsilon, x)$ increases as x increases. and $\to \infty$ as $x \to \infty$. So that it is not possible to choose any number m such that

$\forall n \geq m$ and $\forall x \geq 0$.

So, the convergence is not uniform in $[0,\infty)$.

Now, we consider the interval $[0,k]$.

Let m be a integer greater than $k(\frac{1}{\in}-1)$.

We then see that $\forall n \geq m$ and $\forall x \in [0,k]$, $|f_n(x)-f(x)| < \in$.

Hence, the convergence is uniform in $[0,k]$.

Ex: Show that if $f_n(x) = nxe^{-nx^2}$ the sequence $\{f_n\}$ is point-wise, but not uniformly convergent in $[0,a]$, $a > 0$ i.e. in $[0,\infty)$.

Solution : Here $f_n(x) = nxe^{-nx^2}$.

Now, $e^{nx^2} = 1 + nx^2 + \frac{n^2x^4}{2} + \cdots\cdots > \frac{n^2x^4}{2}$.

$\therefore 0 < nxe^{-nx^2} < \frac{2x}{nx^3}, x > 0.$

By Sandwich theorem, we have

$$\lim nxe^{-nx^2} = 0.$$

$\therefore f(x) = 0, \ \forall x \in [0,\infty).$

This shows that the sequence $\{f_n\}$ is point-wise convergent in $[0,\infty)$ to the function f, where $f(x) = 0$, $\forall x \in [0,\infty)$.

If possible let the sequence be uniformly convergent in $[0,\infty)$. So that for given $\in > 0$, $\exists$ a positive integer m such that

let, ~~$\forall n \geq m$ and~~ $\forall x \geq 0$, $|f_n(x)-f(x)| = nxe^{-nx^2} < \in$. .... (1)

Let m_0 be an integer greater than m and $e^2 \in^2$ and let $x = \frac{1}{m_0}$.

Then, the inequality (1) holds for $x = \frac{1}{\sqrt{m_0}}$ and $n = m_0$.

These give, $\frac{\sqrt{m_0}}{e} < \in \Leftrightarrow m_0 < e^2 \in^2$.

So that we arrive at a contradiction.

Thus, the convergence is not uniform in $[0,\infty)$.

Since $nxe^{-nx^2} < \frac{2}{nx^3}$

Now, $\frac{2}{nx^3} < \epsilon$

$\Rightarrow n > \frac{2}{x^3\epsilon}$

Let $K = [\frac{2}{x^3\epsilon}] + 1$

$\therefore |f_n(x) - f(x)| < \epsilon$, $\forall n \geq K$ & $x \in [0,\infty)$

But as $x \to 0$ then $K \to \infty$

Thus f_n is not uniform convergent

Note1: Choice of $x = \frac{1}{m_0}$ is admissible because the interval contains the origin.

Note2: The interval of uniform convergence is always to be closed interval i.e. it must include the end points but the interval for point-wise or absolutely convergence can be of any type.

Ex: For each $n \in N$, let $f_n(x) = nxe^{-nx^2}$, $x \in [0,1]$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Solution: Since, $0 \le x \le 1$, so we must have

$$e^{nx^2} \ge \frac{n^2x^4}{2} \ge 0$$

$$Or, 0 \le nxe^{-nx^2} \le \frac{2}{nx^3}$$

$\therefore$ By Sandwich Theorem, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = 0$$

Thus, the sequence $\{f_n\}$ converges point-wise on $[0,1]$ to the function f, where

$f(x) = 0, \ \forall x \in [0,1]$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} nxe^{-nx^2}, \quad 0 \le x \le 1$ .... (1)

Let $u(x) = nxe^{-nx^2}$.

$$\therefore u'(x) = ne^{-nx^2} - 2nx.nxe^{-nx^2} = \frac{n - 2n^2x^2}{e^{nx^2}}.$$

Since, $u'(x) = 0$ at $x = \frac{1}{\sqrt{2n}}$.

$u'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2n}}$.

and $u'(x) < 0$ for $x > \frac{1}{\sqrt{2n}}$.

$\therefore u(x)$ is an increasing function on $0 < x < \frac{1}{\sqrt{2n}}$ and decreasing for $x > \frac{1}{\sqrt{2n}}$.

$\therefore$ Maximum value of $u(x)$ is at $x = \frac{1}{\sqrt{2n}}$.

$$\text{So, } \sup_{x\in[0,1]} u(x) = u\left(\frac{1}{\sqrt{2n}}\right) = n\frac{1}{\sqrt{2n}}\cdot\frac{1}{e^{n\cdot\frac{1}{2n}}} = \frac{1}{\sqrt{2}}\sqrt{\frac{n}{e}}.$$

From (1), we have $M_n = \sqrt{\frac{n}{2e}}$.

$$\therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sqrt{\frac{n}{2e}} = \infty \neq 0.$$

Hence, by M_n Test the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Ex: A sequence of function $\{f_n\}$ is defined by $f_n(x) = \dfrac{nx}{1+n^2x^2}, 0 \le x \le 1$.

Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Solution : Since, $f_n(x) = \dfrac{nx}{1+n^2x^2}, \quad 0 \le x \le 1.$

For $x = 0$, the sequence is $\{0,0,\ldots\ldots\ldots\}$ which converges to 0.

For $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = 0.$

Thus the limit function is $f(x) = 0, \ 0 \le x \le 1$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$ ·

$$= \sup_{x\in[0,1]} \frac{nx}{1+n^2x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now, for $x > 0$, we have

$$\frac{\frac{1}{nx} + nx}{2} \ge \sqrt{\frac{1}{nx}.nx}$$, the equality holds for $x = \frac{1}{n}$.

Or, $\dfrac{1+n^2x^2}{nx} \ge 2$ Or, $\dfrac{nx}{1+n^2x^2} \le \dfrac{1}{2}$

So, $\dfrac{nx}{1+n^2x^2} = \dfrac{1}{2}$, for $x = \dfrac{1}{n}$. $\quad \therefore \sup_{x\in[0,1]} \dfrac{nx}{1+n^2x^2} = \dfrac{1}{2}$.

So, from (1), we have

$$M_n = 1/2 \qquad \therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty} 1/2 = 1/2 \neq 0.$$

Hence by $M_n - test$ the given sequence is not uniformly convergent on $[0,1]$.

Another 2nd Process:

Let $|f_n(x) - f(x)| < \epsilon$

$\Rightarrow \dfrac{nx}{1+n^2x^2} < \epsilon$

Now, $\dfrac{nx}{1+n^2x^2} < \dfrac{1}{nx}$

So, $\dfrac{1}{nx} < \epsilon \Rightarrow n > \dfrac{1}{x\epsilon}$

Let, $k = \left[\dfrac{1}{x\epsilon}\right] + 1$

Then, $|f_n(x) - f(x)| < \epsilon$, $\forall n \ge k$

But as $x \to 0$, $k \to \infty$

Thus f_n is not uniform convergent on $[0,1]$.

Ex: For each $n \in N$, let $f_n(x) = 1 - \dfrac{x^n}{n}, \ x \in [0,1]$.

Show that the sequence $\{f_n\}$ is uniformly convergence on $[0,1]$.

Solution : Since, for $0 \le x \le 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(1 - \dfrac{x^n}{n}\right) = 1.$

So, the sequence of function $\{f_n\}$ converges point-wise on $[0,1]$ to a function f

, where $f(x) = 1, \ 0 \le x \le 1$

Let $M_n = \sup_{x\in[0,1]}|f_n(x)-f(x)| = \sup_{x\in[0,1]}\frac{|x^n|}{n} = \sup_{x\in[0,1]}\frac{|x|^n}{n} = \frac{1}{n}$.

$\therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty}\frac{1}{n} = 0.$

Hence by M_n Test the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

Ex: Let $f_n(x) = x^2e^{-nx}$, $x\in[0,\infty)$.

Show that the sequence $\{f_n\}$ is uniformly convergent on $[0,\infty)$.

Solution : Now, $e^{nx} = 1+nx+\frac{n^2x^2}{2!}+\frac{n^3x^3}{3!}+\ldots\ldots\ \geq \frac{n^2x^2}{2} \geq 0.$

So, for $x\geq 0$, $0 \leq x^2e^{-nx} \leq \frac{2}{n^2}$.

$\therefore$ By Sandwich theorem, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^2e^{-nx} = 0,\ \forall x\geq 0.$$

$\therefore$ The limit function is $f(x)=0,\ 0\leq x<\infty$.

Let $M_n = \sup_{x\in[0,\infty)}|f_n(x)-f(x)| = \sup_{x\in[0,\infty)} x^2e^{-nx}$ .... (1)

Let $u(x) = x^2e^{-nx},\ x\geq 0$.

$\therefore\ u'(x) = 2xe^{-nx} - nx^2e^{-nx} = \frac{x(2-nx)}{e^{nx}}$.

So, $u'(x)=0$ at $x=\frac{2}{n}$.

$u'(x)>0$ for $0<x<\frac{2}{n}$

$u'(x)<0$ for $x>\frac{2}{n}$

2nd Process:
Let us choose $\epsilon > 0$.
Then $|f_n(x) - f(x)| < \epsilon$
$\Rightarrow x^2e^{-nx} < \epsilon$
Now, $x^2e^{-nx} < \frac{2}{n^2}$
$\therefore \frac{2}{n^2} < \epsilon \Rightarrow n > \sqrt{2/\epsilon}$
Let $K = [\sqrt{\frac{2}{\epsilon}}]+1$
Thus $|f_n(x) - f(x)| < \epsilon,\ \forall n \geq K(\epsilon)$
This proves that f_n is uniformly convergent on $[0,\infty)$

$\therefore$ u(x) is an increasing function on $0<x<\frac{2}{n}$ and u(x) at maximum at $x=\frac{2}{n}$.

Also, $u(x)$ is a decreasing function for $x>\frac{2}{n}$.

And $\lim_{n\to\infty} u(x) = 0$. $\quad \because \sup_{x\in[0,1]} u(x) = u\left(\frac{2}{n}\right) = \frac{4}{n^2e^2}$.

From (1), we have

$M_n = \frac{4}{n^2e^2}$. $\quad \therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty}\frac{4}{n^2e^2} = 0.$

Hence, by M_n – *Test* the given sequence is uniformly convergent on $[0,\infty)$.

Ex: Let $Q = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\}$ be a countable set of rational numbers in the close interval $[0,1]$ and let $f_n : [0,1] \to R$ be defined by

$$f_n(x) = \begin{cases} 1, & for\ x = x_1, x_2, \ldots\ldots, x_n \\ 0, & elsewhere \end{cases}$$

Let $\{f_n\}_n$ converges point-wise to f.

State with reasons

(*i*) whether each f_n is R-integrable on the interval $[0,1]$.

(*ii*) whether f is R-integrable on the interval $[0,1]$.

(*iii*) whether $\{f_n\}_n$ is uniformly convergent on the interval $[0,1]$. [C.H.'98, 04]

Solution : (*i*) Since $f_n : [0,1] \to R$ be defined as

$$f_n(x) = \begin{cases} 1, & for\ x = x_1, x_2, \ldots\ldots, x_n \\ 0, & elsewhere \end{cases}$$

we see that each f_n is continuous on $[0,1]$ except at $x = x_1, x_2, \ldots\ldots, x_n$.

Since the set of discontinuous points $\{x_1, x_2, \ldots\ldots, x_n\}$ are finite. So has finite number of discontinuity on $[0,1]$.

Thus, each f_n is continuous on $[0,1]$ except at finite number of points and hence each is R-integrable on $[0,1]$.

(*ii*) Since $Q = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\ldots\}$

Now, $\lim_{n\to\infty} f_n(x) = 1$, if $x \in [0,1] \cap Q$

$= 0$, $x \in [0,1] - Q$

∵ The sequence $\{f_n\}$ converges to f on $[0,1]$, where

$f(x) = 1$, if $x \in [0,1] \cap Q$

$= 0$, $x \in [0,1] - Q$

Since, f is discontinuous at every point in $[0,1]$.

∴ The number of discontinuous points are infinite on $[0,1]$.

So, f is not R-integrable on $[0,1]$.

(*iii*) Since each f_n is R-integrable on $[0,1]$ but the limit function f is not R-integrable on $[0,1]$.

∴ The sequence $\{f_n\}$ is not uniformly convergent to f on $[0,1]$.

Ex: Show that the sequence $\{f_n\}$ where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on $[0,1]$.

Solution : For $x = 0$, $f_n(x) = 0$, $\forall n \in N$

For $x = 1$, $f_n(x) = 0$, $\forall n \in N$

For $0 < x < 1$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \quad (\infty/\infty \text{ form})$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} = \lim_{n\to\infty} \frac{-x(1-x)^{+n}}{\log(1-x)} \quad \text{(by L'Hospital Rule)}$$

$$= 0, \quad \text{as } 0 < x < 1. \text{ i.e., } 0 < 1-x < 1$$

Thus the sequence $\{f_n\}$ converges point-wise to the function f on $[0,1]$, where $f(x) = 0$, $\forall x \in [0,1]$.

Let $M_n = \{|f_n(x) - f(x)| : x \in [0,1]\} = \sup_{x\in[0,1]} nx(1-x)^n$ .... (1)

Let $u(x) = nx(1-x)^n$, $0 \le x \le 1$.

$\because u'(x) = n(1-x)^n - n^2 x(1-x)^{n-1} = n(1-x)^{n-1}\{1-(n+1)x\}$.

Since, $u'(x) = 0$ at $x = \frac{1}{n+1}$.

$u'(x) > 0$ for $0 < x < \frac{1}{n+1}$.

and $u'(x) < 0$ for $x > \frac{1}{n+1}$.

$\therefore u(x)$ is an increasing function on $0 < x < \frac{1}{n+1}$ and maximum value of u(x) occur at $x = \frac{1}{n+1}$.

Also, u(x) is decreasing function for $x > \frac{1}{n+1}$ and $\lim_{n\to\infty} u(x) = 0$ as $0 \le x \le 1$.

$$\therefore \sup_{x\in[0,1]} u(x) = u\left(\frac{1}{n+1}\right) = \frac{n}{n+1}\left(1-\frac{1}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}.$$

$\therefore$ From (1), we have

$$M_n = \left(1-\frac{1}{n+1}\right)^{n+1}. \qquad \therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = e^{-1} \neq 0.$$

Hence by M_n test the given sequence is not uniform by convergent on $[0,1]$.

Ex: Let $f_n(x)=ax+\frac{bx^n}{n}$, $a,b\in R, b\neq 0$, $x\in[0,1]$. Show that $\{f_n\}_n$ is uniformly convergent on $[0,1]$ although $\{f_n'\}_n$ is not uniformly convergent on $[0,1]$.

Solution: Since $\lim_{n\to\infty} f_n(x)=ax,\ x\in[0,1]$.

Thus, $\{f_n\}_n$ converges pointwise to f on $[0,1]$

, where $f(x)=ax,\ x\in[0,1]$.

Let $M_n=\sup\{|f_n(x)-f(x)|:\ x\in[0,1]\}=|b|\sup\left\{\frac{x^n}{n}:\ x\in[0,1]\right\}.=\frac{|b|}{n}\to 0$ as $n\to\infty$.

Hence $\{f_n\}_n$ is uniformly convergent to f on $[0,1]$.

Now, $f_n'(x)=a+bx^{n-1},\ x\in[0,1], n\in N.$ $\Rightarrow \lim_{n\to\infty} f_n'(x)=\begin{cases} a, & x\in[0,1) \\ a+b, & x=1. \end{cases}$

Since limit function of $\{f_n'\}_n$ is not continuous on $[0,1]$, although for each $n\in N$, f_n' is continuous on $[0,1]$.

This shows that $\{f_n'\}_n$ is not uniformly convergent on $[0,1]$.

Ex: Show that the function $\{f_n\}_n$ defined by

$$f_n(x)=n^2x^n,\ x\in[0,1)$$
$$=1,\quad x=1.$$

is not uniformly convergent on $[0,1]$.

Solution: Since $f_n(0)=0,\ f_n(1)=1,\ \forall n\in N.$

$$\Rightarrow \lim_{n\to\infty} f_n(0)=0,\ \lim_{n\to\infty} f_n(1)=1.$$

For, $0<x<1$, *let* $x=\frac{1}{1+a}$ $(a>0)$.

Mind it Then, $f_n(x)=\frac{n^2}{(1+a)^n}<\frac{n^2}{\frac{n(n-1)(n-2)}{3!}a^3}=\frac{6n}{(n-1)(n-2)a^3}$.

$$\Rightarrow 0<f_n(x)<\frac{6}{\left(1-\frac{1}{n}\right)(n-2)a^3},\quad 0<x<1 \text{ and } n>2.$$

By Sandwich theorem, we have

$\lim_{n\to\infty} f_n(x) = 0, \quad 0 < x < 1.$

Thus the sequence $\{f_n\}_n$ converges point wise to f on $[0,1]$

; where $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$

Since the limit function f is not continuous on $[0,1]$, so $\{f_n\}$ is not uniformly convergent on $[0,1]$. **(Proved)**

Ex: Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ is uniformly convergent in $[0, \pi]$.

Solution : Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{\sin nx}{\sqrt{n}} = 0.$.

Let $M_n = \sup_{x\in[0,\pi]} |f_n(x) - f(x)| = \sup_{x\in[0,\pi]} \left|\dfrac{\sin nx}{\sqrt{n}}\right|$ ....(1)

Let $y = \dfrac{\sin nx}{\sqrt{n}}$. $\therefore \dfrac{dy}{dx} = \sqrt{n}\cos nx$.

For maximum and minimum value of y, $\dfrac{dy}{dx} = 0$

$Or,\ \cos nx = 0 = \cos\dfrac{\pi}{2}.$ $Or,\ nx = \dfrac{\pi}{2}.$ $Or, x = \dfrac{\pi}{2n} \in [0, \pi].$

Again, $\dfrac{d^2y}{d^2x} = -n^{\frac{3}{2}}\sin nx$. $\therefore \left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{\pi}{2n}} = -n^{\frac{3}{2}}\sin\left(\dfrac{\pi}{2}\right) = -n^{\frac{3}{2}} < 0.$

So, y is maximum when $x = \dfrac{\pi}{2n}$ and maximum value of $y = \dfrac{\sin\frac{\pi}{2}}{\sqrt{n}} = \dfrac{1}{\sqrt{n}}$.

$\because \sup_{x\in[0,\pi]} y = \dfrac{1}{\sqrt{n}}$.

From (1), we have

$$M_n = \frac{1}{\sqrt{n}}. \quad \therefore \lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Hence by M_n test the given sequence is uniformly convergent on $[0, \pi]$.

Theorem : Let [a, b] be a closed bounded interval and for each $n \in N$, $f_n : [a,b] \to R$ be bounded on [a,b]. If the sequence $\{f_n\}$ be uniformly convergent on [a,b], then the limit function f is bounded on [a, b]. **[V.H'12]**

Proof : Let $\in = 0\cdot001$

Since $\{f_n\}$ is uniformly convergent on [a,b] to f (limit function).

Then $\exists$ is a natural number k such that

$\forall x \in [a,b], |f_n(x) - f(x)| < 0\cdot001, \forall n \geq k$

$\therefore \forall x \in [a,b], |f_k(x) - f(x)| < 0\cdot001$ ---------- (1)

Also, f_k is bounded on [a,b], then $\exists$ a positive real number B such that

$|f_k(x)| \leq B, \forall x \in [a,b]$ ---------- (2)

Now, $\forall x \in [a,b]$

$|f(x)| = |f(x) - f_k(x) + f_k(x)|$

$\leq |f_k(x) - f(x)| + |f_k(x)| < 0\cdot001 + B$, by (1) and (2)

This proves that f is bounded on [a, b].

Note : If such f_n is bounded on [a, b], the uniform convergence of $\{f_n\}$ on [a, b] is a sufficient but not a necessary condition for bounded ness of the limit function f on [a, b].

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in [0,1]$.

Then the limit function f is defined by $f(x) = 0$, $0 \leq x \leq 1$.

Since, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = \frac{1}{2}$

$\because M_n = \frac{1}{2}$. Or, $\lim_{n\to\infty} M_n = \frac{1}{2} \neq 0$.

Since $\sup_{x\in[0,1]} |f_n(x)| = \frac{1}{2}$, each f_n is bounded on [0, 1]. Also the limit function f is bounded on [0,1]. But the convergence is not uniform on [0,1].

Theorem: If the sequences $\{f_n\}$ and $\{g_n\}$ converges uniformly to f and g respectively on $X \subseteq R$ and for each $n \in N$, both f_n and g_n are bounded on X, then show that the sequence $\{f_n g_n\}$ converges uniformly to fg on X.

Proof: Since for each $n \in N$, f_n, g_n are bounded on X and $\{f_n\}$, $\{g_n\}$ converges uniformly there, so f_n, g_n are uniformly bounded on X and f, g are also bounded.

Then there exists positive numbers K_1 & K_2 such that