Semester-II

Course Type- core-3

Course Title-C3T: Real Analysis

Topic- Sequence of Real Number

References— S.K. Mapa book.

Date: 29.03.2020

—Shambhu Nath Acharya

Proof: Let $\varepsilon > 0$. It follows from the convergence of the sequences $\{u_n\}$ and $\{w_n\}$ that there exists natural numbers k_1 and k_2 such that $|u_n - l| < \varepsilon$ for all $n \geq k_1$ and $|w_n - l| < \varepsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $l - \varepsilon < u_n < l + \varepsilon$ and $l - \varepsilon < w_n < l + \varepsilon$ for all $n \geq k$

Then $l - \varepsilon < u_n \leq v_n \leq w_n < l + \varepsilon$ for all $n \geq k$

Consequently, $|v_n - l| < \varepsilon$ for all $n \geq k$

This shows that the sequence $\{v_n\}$ is convergent and $\lim\limits_{n\to\infty} v_n = l$.

Ex 8: Prove that $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots\dots\dots\dots + \frac{1}{\sqrt{n^2+n}}\right) = 1$. **VU'2003**

Let $u_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots\dots\dots + \frac{1}{\sqrt{n^2+n}}$.

We have $\frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$$

....

$$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}.$$

Therefore $u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$

Again, $\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$

$$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$$

....

Therefore $u_n > \frac{n}{\sqrt{n^2+n}}$ for all $n \geq 2$

Thus $\frac{n}{\sqrt{n^2+n}} < u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$

But $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$ and $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$.

By Sandwich theorem, $\lim\limits_{n\to\infty} u_n = 1$

Ex 9: Use sandwich theorem to prove that (i) $\lim\limits_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0$

(ii) $\lim\limits_{n\to\infty}\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots\dots\dots\dots + \frac{1}{(n+n)^2}\right] = 0$.

Let $v_n = \sqrt{n+1} - \sqrt{n}$ for all $n \in \mathbb{N}$.

Then $v_n = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$ for all $n \in \mathbb{N}$

Therefore $\dfrac{1}{2\sqrt{n+1}} < \dfrac{1}{\sqrt{n+1}+\sqrt{n}} < \dfrac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$

Let $u_n = \dfrac{1}{2\sqrt{n+1}}$ and $w_n = \dfrac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$.

Then we have, $u_n < v_n < w_n$ for all $n \in \mathbb{N}$.

$\therefore$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{2\sqrt{n+1}} = 0$ and $\lim\limits_{n\to\infty} w_n = \lim\limits_{n\to\infty} \dfrac{1}{2\sqrt{n}} = 0$.

Therefore we have $\lim\limits_{n\to\infty} v_n = 0$, i.e., $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = 0$.

(ii) Try your self

Null Sequence: A sequence $\{u_n\}$ is said to be a null sequence if $\lim\limits_{n\to\infty} u_n = 0$.

Ex 10: Show that the sequence $\left\{\dfrac{1}{n}\right\}$ is a null sequence.

Let us choose a positive ε.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $k\varepsilon > 1$, i.e., $0 < \dfrac{1}{k} < \varepsilon$. This implies $0 < \dfrac{1}{n} < \varepsilon$ for all $n \ge k$.

It follows that $\left|\dfrac{1}{n} - 0\right| < \varepsilon$ for all $n \ge k$.

This proves $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Therefore the sequence $\left\{\dfrac{1}{n}\right\}$ is a null sequence.

Ex 11: Show that the sequence $\left\{\dfrac{1}{n^p}\right\}$, where $p > 0$, is a null sequence.

Let $\varepsilon > 0$

Then $\left|\dfrac{1}{n^p} - 0\right| = \dfrac{1}{n^p} < \varepsilon$ if $n^p > \dfrac{1}{\varepsilon}$ i.e., if $n > \left(\dfrac{1}{\varepsilon}\right)^{\frac{1}{p}}$.

Let $k = \left[\left(\dfrac{1}{\varepsilon}\right)^{\frac{1}{p}}\right] + 1$.

Then k is a natural number and $\left|\dfrac{1}{n^p} - 0\right| < \varepsilon$ for all $n \ge k$.

This proves that $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$. i.e., $\left\{\dfrac{1}{n^p}\right\}$, $p > 0$ is a null sequence.

Ex 12: Show that $\lim\limits_{n\to\infty} r^n = 0$ if $|r| < 1$.

Case1: $r=0$. In this case the sequence is $\{0,0,0,\ldots\ldots\ldots\ldots\}$.

The sequence converges to 0.

That is, $\lim_{n\to\infty} r^n = 0$ when $r=0$.

Case2: $r \neq 0$ and $|r|<1$.

$\frac{1}{|r|} > 1$, since $|r|<1$. Let $\frac{1}{|r|} = a+1$ where $a>0$.

$$|r^n - 0| = |r^n| = |r|^n = \frac{1}{(a+1)^n}.$$

We have $(1+a)^n > na$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Then $|r^n - 0| < \varepsilon$ holds if $n > \frac{1}{a\varepsilon}$

Let $k = \left[\frac{1}{a\varepsilon}\right]+1$. Then k is a natural number and $|r^n - 0| < \varepsilon$ for all $n \geq k$.

Since ε is arbitrary, $\lim_{n\to\infty} r^n = 0$.

Combining the cases, $\lim_{n\to\infty} r^n = 0$ if $|r|<1$.

Ex 13: Let $\{u_n\}$ be a bounded sequence and $\lim_{n\to\infty} v_n = 0$. Prove that $\lim_{n\to\infty} u_n v_n = 0$.

Utilize this to prove that $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$. **CU'2001, 06**

Since $\{u_n\}$ is a bounded sequence, there exists a positive real number B such that $|u_n| < B$ for all $n \in \mathbb{N}$.

Let us choose $\varepsilon > 0$.

Since $\lim_{n\to\infty} v_n = 0$, there exists a natural number k such that $|v_n - 0| < \frac{\varepsilon}{B}$ for all $n \geq k$

Now, $|u_n v_n - 0| = |u_n||v_n| < \frac{\varepsilon}{B} B = \varepsilon$ for all $n \geq k$

This implies that $\lim_{n\to\infty} u_n v_n = 0$.

2nd Part: Let $u_n = (-1)^n$ and $v_n = \frac{n}{n^2+1}$ for all $n \in \mathbb{N}$.

Then $|u_n| = 1 < 2$ for all $n \in \mathbb{N}$

This implies that $\{u_n\}$ is a bounded sequence

Now, $\frac{1}{2(n^2+1)} < v_n < \frac{1}{n}$ for all $n \geq 1$ and $\lim_{n\to\infty} \frac{1}{2(n^2+1)} = 0, \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore by Sandwich theorem we have, $\lim_{n\to\infty} v_n = 0$

Hence by above result, $\lim_{n\to\infty} u_n v_n = 0$, i.e., $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$.

Theorem: If $\{u_n\}$ be null sequence then $\{|u_n|\}$ is a null sequence and conversely.

Proof: Let $\varepsilon > 0$. Since $\lim_{n\to\infty} u_n = 0$, there exists a natural number k such that $|u_n - 0| < \varepsilon$ for all $n \ge k$

As $||u_n| - 0| = |u_n|$, it follows that $||u_n| - 0| < \varepsilon$ for all $n \ge k$

This proves that $\lim_{n\to\infty} |u_n| = 0$.

Conversely, let $\lim_{n\to\infty} |u_n| = 0$.

Let $\varepsilon > 0$. There exists a natural number k such that $||u_n| - 0| < \varepsilon$ for all $n \ge k$.

That is, $|u_n| < \varepsilon$ for all $n \ge k$.

This proves that $\lim_{n\to\infty} u_n = 0$.

Divergent sequence: A real sequence $\{u_n\}$ is said to diverge to ∞ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that $u_n > G$ for all $n \ge k$.

In this case we write $\lim_{n\to\infty} u_n = \infty$.

A real sequence $\{u_n\}$ is said to diverge to $-\infty$ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that $u_n < -G$ for all $n \ge k$.

In this case we write $\lim_{n\to\infty} u_n = -\infty$.

A real sequence $\{u_n\}$ is said to be a properly divergent sequence if it either diverges to ∞, or diverges to $-\infty$.

Theorem: A sequence $\{u_n\}$ diverging to ∞ is unbounded above but bounded below.

Proof: Let a sequence $\{u_n\}$ diverges to ∞. Then for each pre-assigned positive number G there exists a natural number k such that $u_k > G$.

Therefore there does not exist a real number B such that $u_n \le B$ holds for all $n \in \mathbb{N}$. In other words, $\{u_n\}$ is unbounded above.

Let $G > 0$. Then there exists a natural number k such that $f(n) > G$ for all $n \ge k$.

Let $b = \min\{u_1, u_2, \dots\dots\dots, u_{k-1}, G\}$. Then $u_n \ge b$ for all $n \in \mathbb{N}$

This proves that the sequence $\{u_n\}$ is bounded below.

Note: A sequence unbounded above but bounded below may not diverge to ∞.

For example, let us consider the sequence $\{u_n\}$ where $u_n = n^{(-1)^n}$.

The sequence is $\left\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \dots\dots\dots\dots\dots\right\}$. The sequence is unbounded above and bounded below, 0 being a lower bound. The sequence does not diverge to ∞, because for a pre-assigned positive number G there does not exist a natural number k such that $u_n > G$ holds for all $n \ge k$.

Theorem: A sequence diverging to $-\infty$ is unbounded below but bounded above.

VU'2001

(H.W.)

Note: A sequence unbounded below but bounded above may not diverge to $-\infty$.

Definition: A bounded sequence that is not convergent is said to be an **oscillatory sequence of finite oscillation.**

An unbounded sequence that is not properly divergent is said to be an **oscillatory sequence of infinite oscillation.**

An oscillatory sequence is therefore neither convergent nor properly divergent sequence. It is called an improperly **divergent sequence.**

Examples:

1: The sequence $\{2^n\}$ diverges to ∞.

2: The sequence $\{-n^2\}$ diverges to $-\infty$.

3: The sequence $\{(-1)^n\}$ is a bounded sequence, but not convergent. It is an oscillatory sequence if finite oscillation.

4: The sequence $\{(-1)^n n\}$ is an unbounded sequence, and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

Behavior of the sequence $\{r^n\}$ for different values of r:

Case1: $r>1$. Let $r=1+a$ where $a>0$.

Then $r^n=(1+a)^n>1+na$ for $n>1$

Let $G>0$. Then $1+na>G$ holds if $n>\dfrac{G-1}{a}$

Let $k=\left[\dfrac{G-1}{a}\right]+1$. Then k is a natural number and $r^n>G$ for all $n\geq k$.

Since G is an arbitrary positive number, $\lim\limits_{n\to\infty} r^n=\infty$

Therefore in this case the sequence diverges to ∞.

Case2: $r=1$. In this case the sequence is $\{1,1,1,............\}$ and the sequence converges to 1.

Case3: $|r|<1$. In this case the sequence converges to 0.

Case4: $r=-1$. In this case the sequence is $\{-1,1,-1,1,..................\}$. The sequence is bounded but not convergent. The sequence is an oscillatory sequence of finite oscillation.

Case5: $r<-1$. Let $r=-s$. Then $s>1$

The sequence is $\{(-1)^n s^n\}$. It is an unbounded sequence. It neither diverges to ∞ nor diverges to $-\infty$. It is an oscillatory sequence of infinite oscillation.

Theorem: Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=l$.

(i) If $0\leq l<1$ then $\lim\limits_{n\to\infty} u_n=0$,

(ii) (ii) if $l>1$ then $\lim\limits_{n\to\infty} u_n=\infty$.

Proof: (i) Let us choose a positive ε such that $l+\varepsilon<1$.

Since $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=l$, there exists a natural number k

Such that $l-\varepsilon<\dfrac{u_{n+1}}{u_n}<l+\varepsilon$ for all $n\geq k$.

Let $l+\varepsilon=r$. Then $0<r<1$

Therefore $\frac{u_{n+1}}{u_n} < r$ for all $n \geq k$

since 5+1, 5+2, 5+3, 5+4, 5+5, 5+6 (11) ∴ n = 11 − 5 = 6

Hence we have $\frac{u_{k+1}}{u_k} < r$, $\frac{u_{k+2}}{u_{k+1}} < r$, ..., $\frac{u_n}{u_{n-1}} < r$ for $n \geq k+1$.

Multiplying, $\frac{u_n}{u_k} < r^{n-k}$ for $n \geq k+1$

Or, $u_n < \frac{u_k}{r^k} r^n$ for $n \geq k+1$

Now $\lim_{n\to\infty} r^n = 0$ since $0 < r < 1$; and $\frac{u_k}{r^k}$ is a fixed positive number. ∴ $0 < u_n < r^n \Rightarrow 0 < u_n < 0$ by sandwich theorem.

Therefore $\lim_{n\to\infty} u_n = 0$

(ii) Let us choose a positive number ε such that $l - \varepsilon > 1$.

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, there exists a natural number m

Such that $l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon$ for all $n \geq m$.

Let $l - \varepsilon = s$. Then $s > 1$

Therefore $\frac{u_{n+1}}{u_n} > s$ for all $n \geq m$

Hence we have $\frac{u_{m+1}}{u_m} > s$, $\frac{u_{m+2}}{u_{m+1}} > s$, ..., $\frac{u_n}{u_{n-1}} > s$ for $n \geq m+1$.

Multiplying, $\frac{u_n}{u_m} > s^{n-m}$ for $n \geq m+1$

Or, $u_n > \frac{u_m}{s^m} s^n$ for $n \geq m+1$

Now $\lim_{n\to\infty} s^n = \infty$ since $s > 1$; and $\frac{u_m}{s^m}$ is a fixed positive number.

Therefore $\lim_{n\to\infty} u_n = \infty$

Note: If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, no definite conclusion can be made about the nature of the sequence. For example, (i) if $u_n = \frac{n+1}{n}$ then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ and $\lim_{n\to\infty} u_n = 1$; (ii) if $u_n = \frac{1}{n}$ then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ and $\lim_{n\to\infty} u_n = 0$.

Theorem: Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} u^{\frac{1}{n}} = l$.

(i) If $0 \leq l < 1$ then $\lim_{n\to\infty} u_n = 0$

(ii) If $l > 1$ then $\lim_{n\to\infty} u_n = \infty$.

Proof: (i) Let us choose a positive ε such that $l + \varepsilon < 1$.

Since $\lim_{n\to\infty} u^{\frac{1}{n}} = l$, there exists a natural number k

Such that $l-\varepsilon < u_n^{\frac{1}{n}} < l+\varepsilon$ for all $n \geq k$.

Let $l+\varepsilon = r$. Then $0 < r < 1$ and $u_n^{\frac{1}{n}} < r$ for all $n \geq k$

So we have $0 < u_n < r^n$ for all $n \geq k$.

Since $\lim_{n\to\infty} r^n = 0$, $\lim_{n\to\infty} u_n = 0$, by Sandwich theorem.

(ii) Let us choose a positive number ε such that $l-\varepsilon > 1$.

Since $\lim_{n\to\infty} u_n^{\frac{1}{n}} = l$, there exists a natural number m

Such that $l-\varepsilon < u_n^{\frac{1}{n}} < l+\varepsilon$ for all $n \geq m$.

Let $l-\varepsilon = s$. Then $s > 1$ and $u_n^{\frac{1}{n}} > s$ for all $n \geq m$

So we have $u_n > s^n$ for all $n \geq m$.

Since $s > 1$, $\lim_{n\to\infty} s^n = \infty$ and therefore $\lim_{n\to\infty} u_n = \infty$

Note: If $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$, no definite conclusion can be made about the nature of the sequence $\{u_n\}$.

For example, (i) if $u_n = \dfrac{n+1}{n}$ then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$ and $\lim_{n\to\infty} u_n = 1$; (ii) if $u_n = \dfrac{n+1}{2n}$ then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$ and $\lim_{n\to\infty} u_n = \dfrac{1}{2}$.

Ex 14: Show that $\left\{\dfrac{x^n}{n!}\right\}$, $x \in \mathbb{R}$ is a null sequence

✍ Let $u_n = \dfrac{x^n}{n!}$ for all $n \geq 1$ and $x \in \mathbb{R}$.

Then $\left|\dfrac{u_{n+1}}{u_n}\right| = \dfrac{1}{n+1}|x| \to 0$ as $n \to \infty$

This implies that $\lim_{n\to\infty} u_n = 0$. That is, $\left\{\dfrac{x^n}{n!}\right\}$, $x \in \mathbb{R}$ is a null sequence.

Ex 15: Show that the sequence $\{nx^n\}$, where $|x| < 1$ is a null sequence.

✍ Let $u_n = nx^n$

Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{(n+1)x^{n+1}}{nx^n}\right| = \lim_{n\to\infty}\left|\left(1+\dfrac{1}{n}\right)x\right| = |x| < 1$

Thus $\lim_{n\to\infty}|u_n| = 0$ i.e. $\{|u_n|\}$ is a null sequence.

Therefore $\{u_n\}$ i.e. $\{nx^n\}$ is a null sequence

Ex 16: Prove that the sequence $\{u_n\}$ where $u_n = \dfrac{n!}{n^n}$ is a null sequence.

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \quad \text{for all } n \in \mathbb{N}.$$

Thus $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{e} < 1$, since $2 < e < 3$

Therefore we have, $\lim\limits_{n\to\infty} u_n = 0$ and this proves that $\{u_n\}$ is a null sequence.

Ex 17: For any real numbers p and $(a > 0)$, prove that $\lim\limits_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$.

Let $u_n = \frac{n^p}{(1+a)^n}$

Then $\frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n}\right)^p}{(1+a)}$

Then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^p}{(1+a)} = \frac{1}{1+a}$

Since $a > 0$ therefore $0 < \frac{1}{1+a} < 1$

Thus $0 < \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} < 1 \Rightarrow \lim\limits_{n\to\infty} u_n = 0$

(19) $\frac{u_{n+1}}{u_n} = \left|\frac{(m-n)x}{n+1}\right| = \left|\frac{\left(\frac{m}{n}-1\right)x}{1+\frac{1}{n}}\right|$ → $|x|$ as $n \to \infty$

Ex 18: Show that $\lim\limits_{n\to\infty} \frac{n^p}{x^n} = 0$, provided $|x| > 1$.

(H.W.)

Ex 19: Show that $\lim\limits_{n\to\infty} \frac{m(m-1)(m-2)................(m-n+1)}{n!} x^n = 0$, if $|x| < 1$ and m, any real number.

(H.W.)

Ex 20: A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for $n \geq 1$ and $0 < u_1 < u_2$. Prove that the sequence $\{u_n\}$ converges to $\frac{u_1 + 2u_2}{3}$. **VU'1997**

$u_2 - u_1 > 0$

$u_3 - u_2 = \frac{1}{2}(u_2 + u_1) - u_2 = -\frac{1}{2}(u_2 - u_1)$

$u_4 - u_3 = \frac{1}{2}(u_3 + u_2) - u_2 = -\frac{1}{2}(u_3 - u_2) = \left(-\frac{1}{2}\right)^2 (u_2 - u_1)$

....

$u_n - u_{n-1} = \left(-\frac{1}{2}\right)^{n-2} (u_2 - u_1)$.

Therefore $u_n - u_1 = (u_2 - u_1)\left[1+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2+\ldots\ldots\ldots\ldots+\left(-\frac{1}{2}\right)^{n-2}\right]$

$$= \frac{2(u_2-u_1)}{3}\left[1-\left(-\frac{1}{2}\right)^{n-1}\right].$$

Now $\lim_{n\to\infty}(u_n - u_1) = \frac{2}{3}(u_2-u_1)$ since $\lim_{n\to\infty}\left(-\frac{1}{2}\right)^{n-1}=0$

Therefore $\lim_{n\to\infty} u_n = u_1 + \frac{2}{3}(u_2-u_1) = \frac{u_1+2u_2}{3}$

Ex 21: Prove that the sequence $\{u_n\}$ defined by $0<u_1<u_2$ and $u_{n+2} = \frac{2u_{n+1}+u_n}{3}$ for $n\geq 1$, converges to $\frac{u_1+3u_2}{4}$

(H.W.)

Ex 22: Prove that the sequence $\{u_n\}$ defined by $0<u_1<u_2$ and $u_{n+2} = \frac{u_{n+1}+2u_n}{3}$ for $n\geq 1$, converges to $\frac{2u_1+3u_2}{5}$

CU'2006

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Ex 23: $0<u_1<u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n\geq 1$, converges to the limit $\sqrt[3]{u_1u_2^2}$

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Ex 24: $0<u_1<u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n\geq 1$, converges to the limit $\frac{3}{\left(\frac{1}{u_1}+\frac{2}{u_2}\right)}$

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(H.W.)

Monotone sequence:

A real sequence $\{u_n\}$ is said to be a **monotone increasing** sequence if $u_{n+1}\geq u_n$ for all $n\in\mathbb{N}$

A real sequence $\{u_n\}$ is said to be a **monotone decreasing** sequence if $u_{n+1}\leq u_n$ for all $n\in\mathbb{N}$.

A real sequence $\{u_n\}$ is said to be a **monotone sequence** if it is either a monotone increasing sequence or a monotone decreasing sequence.

Note: If $u_{n+1}>u_n$ for all $n\in\mathbb{N}$, the sequence $\{u_n\}$ is said to be **strictly monotone increasing** sequence.

If $u_{n+1}<u_n$ for all $n\in\mathbb{N}$, the sequence $\{u_n\}$ is said to be **strictly monotone decreasing** sequence.

If for some natural number m, $u_{n+1}\geq u_n$ for all $n\geq m$ the sequence $\{u_n\}$ is said to be an **'ultimately' monotone increasing** sequence.

If for some natural number m, $u_{n+1}\leq u_n$ for all $n\geq m$ the sequence $\{u_n\}$ is said to be an **'ultimately' monotone decreasing** sequence.

Examples