Semester - IV

Course Type - Core-8

Course Title: C8T: Series of Function

Topic: Series of function

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*** Introduction:

For each $n \in \mathbb{N}$, if $f_n : D \to \mathbb{R}$ be a function defined on a subset D of $\mathbb{R}$ then $\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots$ is an infinite Series of functions defined on D.

Pointwise and uniform convergence:

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ be a function on D. Let for each $n \in \mathbb{N}$, $S_n : D \to \mathbb{R}$ be a function defined by $S_n(x) = \sum_{i=1}^{n} f_i(x)$, $x \in D$. Then

$$S_1(x) = f_1(x),$$
$$S_2(x) = f_1(x) + f_2(x),$$
$$S_3(x) = f_1(x) + f_2(x) + f_3(x),$$
$$\cdots\cdots$$
$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x),$$
$$\cdots\cdots$$

Therefore $\{S_n(x)\}$ is a Sequence of function on D. which is called a Sequence of Partial Sums of the infinite Series $\sum_{n=1}^{\infty} f_n$.

If the Sequence of functions $\{S_n(x)\}$ converges pointwise to a function S on D ⊛ then we say that the infinite Series $\sum_{n=1}^{\infty} f_n$ converges pointwise to S and S is called the Sum function of the Series $\sum_{n=1}^{\infty} f_n$.

If the Sequence $\{S_n(x)\}$ converges uniformly to S, i.e., corresponding to a Preassigned Positive ϵ, there exists a natural number K (depending on ϵ but not on $x \in D$) Such that

for all $x \in D$, $|S_n(x) - S(x)| < \epsilon$, for all $n \geq K$.

Then the infinite Series $\sum_{n=1}^{\infty} f_n$ is Said to converge uniformly to S on D and we write $S = \sum_{n=1}^{\infty} f_n$

⊛ i.e., if for each $x \in D$, the Sequence $\{S_n(x)\}$ converges or $\lim_{n \to \infty} S_n(x) = S(x)$ on D.

Example: Prove that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$, $x \in [0,1]$ is not uniformly convergent on $[0,1]$.

Ans: Let $f_n(x) = \frac{x^2}{(1+x^2)^{n-1}}$, $n \in N$, $x \in [0,1]$.

Let $S_n(x) = \sum_{i=1}^{n} f_i(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

$$= x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^{n-1}}$$

$$= x^2\left[1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^{n-1}}\right]$$

When $x=0$, $S_n(x) = 0$

when $0 < x \leq 1$, $S_n(x) = x^2 . 1 . \dfrac{1-\left(\frac{1}{1+x^2}\right)^n}{1-\frac{1}{1+x^2}}$

$$= 1+x^2 - \frac{1}{(1+x^2)^{n-1}}$$

Thus $\lim_{n\to\infty} S_n(x) = 1+x^2$, $0 < x \leq 1$
$= 0$, $x = 0$

Hence the sequence $\{S_n\}$ converges pointwise to the function S where

$S(x) = 1+x^2$, $0 < x \leq 1$
$= 0$, $x = 0$

Here S is not continuous on $[0,1]$, the point of discontinuity being 0. Each S_n is continuous on $[0,1]$. Therefore the convergence of the sequence $\{S_n\}$ is not uniform on $[0,1]$. Which implies that the series $\sum_{n=1}^{\infty} f_n(x)$ is not uniformly convergent on $[0,1]$.

Absolutely convergent.

The infinite series $\sum_{n=1}^{\infty} f_n$ of functions f_n defined on $D \subset \mathbb{R}$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |f_n|$ is convergent for each $x \in D$.

Theorem: Weierstrass's M-test

Let $D \subset \mathbb{R}$ and $\sum_{n=1}^{\infty} f_n$ be a series of functions on D to $\mathbb{R}$.

Let $\{M_n\}$ be a sequence of positive real numbers such that for all $x \in D$, $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} M_n$ be convergent then the series $\sum_{n=1}^{\infty} f_n$ is uniformly and absolutely convergent on D.

Proof: Let ϵ be any chosen positive number. Since $\sum_{n=1}^{\infty} M_n$ is convergent, by Cauchy's Principle, there exists a natural number K such that

$$|M_{n+1} + M_{n+2} + \cdots + M_{n+p}| < \epsilon \text{ for all } n \geq K,\ p = 1, 2, 3, \cdots$$

For all $x \in D$,

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)|$$
$$\leq |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)|$$
$$\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p} \text{ for all } n \in \mathbb{N}.$$
$$< \epsilon \text{ for all } n \geq K \text{ and } p = 1, 2, 3, \cdots$$
(Since each $M_n > 0$)

$\therefore$ By Cauchy's Principle, $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D.

Again for all $x \in D$,

$$\big||f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)|\big|$$
$$= |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)|$$
$$\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p}$$
$$< \epsilon \text{ for all } n \geq K \text{ and } p = 1, 2, 3, \cdots$$
(Since each $M_n > 0$)

$\therefore$ By Cauchy's Principle, $\sum_{n=1}^{\infty} |f_n|$ is convergent on D. This implies $\sum_{n=1}^{\infty} f_n$ is absolutely convergent on D.

Note: It is a sufficient condition.

Example: Show that the series $\sum_{n=1}^{\infty} \frac{x}{n^p + x^2 n^q}$ converges uniformly for all real x if $p+q>2$.

Ans: Let $f_n(x) = \frac{x}{n^p + x^2 n^q}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$

Then $f(x) = \lim_{n\to\infty} f_n(x) = 0$, $x \in \mathbb{R}$.

For $x \neq 0$, $|f_n(x)| = \frac{|x|}{n^p + x^2 n^q}$.

Now consider two positive numbers $\frac{n^p}{|x|}$, $|x| n^q$ and applying their A.M. $\geq$ G.M.

we get $\frac{\frac{n^p}{|x|} + |x| n^q}{2} \geq \sqrt{\frac{n^p}{|x|} \cdot |x| n^q}$

or, $\frac{n^p + x^2 n^q}{2|x|} \geq \sqrt{n^{p+q}}$

or $\frac{|x|}{n^p + x^2 n^q} \leq \frac{1}{2n^{\frac{p+q}{2}}}$

The equality sign occurs when $\frac{n^p}{|x|} = |x| n^q$

i.e., $|x| = n^{\frac{p-q}{2}}$

Then $|f_n(x)| \leq \frac{1}{2n^{\frac{p+q}{2}}} = M_n$ (say)

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{p+q}{2}}}$ is p-series, so it is convergent if $\frac{p+q}{2} > 1$

Therefore $\sum_{n=1}^{\infty} M_n$ is convergent if $p+q>2$ and $M_n > 0$ for all $n \in \mathbb{N}$.

Thus by Weierstrass's M-test the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent for all $x \in \mathbb{R}$ and if $p+q>2$.

<u>Theorem:</u> Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ is a continuous function on D. If the series $\sum f_n$ be uniformly convergent on D then sum function S is continuous on D.

<u>Ans:</u> Let $S_n(x) = \sum_{i=1}^{n} f_i(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, $x \in D$.

Since for each $n \in \mathbb{N}$, $f_n(x)$ is continuous function on D.

Again sum of finite number of continuous function on D is continuous on D.

So $S_n(x)$ is continuous on D for each $n \in \mathbb{N}$.

Since the series $\sum f_n(x)$ is uniformly convergent on D, the sequence $\{S_n(x)\}$ is uniformly convergent on D to the function $S(x)$.

Let us choose $\epsilon > 0$. Then there exists a natural number K such that

for all $x \in D$, $|S_n(x) - S(x)| < \frac{\epsilon}{3}$ for all $n \geq K$.

It follows that for all $x \in D$, $|S_K(x) - S(x)| < \frac{\epsilon}{3}$ --- (i)

Let $c \in D$, Then (i) gives $|S_K(c) - S(c)| < \frac{\epsilon}{3}$.

Since $S_K(x)$ is continuous at c, there exists a positive δ such that

$$|S_K(x) - S_K(c)| < \frac{\epsilon}{3} \text{ for all } x \in N(c,\delta) \cap D.$$

Thus
$$\begin{aligned}
|S(x) - S(c)| &= |S(x) - S_K(x) + S_K(x) - S_K(c) + S_K(c) - S(c)| \\
&\leq |S(x) - S_K(x)| + |S_K(x) - S_K(c)| + |S_K(c) - S(c)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \text{ for all } x \in N(c,\delta) \cap D.
\end{aligned}$$

This proves that $S(x)$ is continuous at c.
Since c is arbitrary point of D, therefore $S(x)$ is continuous on D.

<u>Note 1.</u> If for each $n \in \mathbb{N}$, $f_n(x)$ is continuous on D and the sum function $S(x)$ of the series $\sum f_n$ is not continuous on D then it follows from the theorem that the convergence of the series $\sum f_n$ is not uniform on D.

<u>Note 2.</u> In this theorem the condition are sufficient but not necessary.

Theorem: (Term-by-Term Integration).

Let $I=[a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, $f_n: I \to \mathbb{R}$ be integrable on I. If the series $\sum_{n=1}^{\infty} f_n$ be uniformly convergent on I to the function S then

(i) S is integrable on I,

(ii) $\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b S(x)\,dx$

Ans: Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$, $x \in [a,b]$, $n \in \mathbb{N}$.

Here $f_n(x)$ be integrable on $[a,b]$, $\forall n \in \mathbb{N}$

Since for each $n \in \mathbb{N}$, $S_n(x)$ is the sum of finite number of integrable functions on $[a,b]$, so $S_n(x)$ is also integrable on $[a,b]$.

Since the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a,b]$ to the function $S(x)$.

$\Rightarrow$ the sequence $\{S_n(x)\}$ is uniformly convergent on $[a,b]$ to $S(x)$.

i.e., for $\epsilon > 0$, there exists a natural number K such that

for all $x \in [a,b]$, $|S_n(x) - S(x)| < \frac{\epsilon}{4(b-a)}$, $\forall n \geq K$.

Let $R_n(x) = S(x) - S_n(x)$, $x \in [a,b]$.

Then $|R_K(x)| < \frac{\epsilon}{4(b-a)}$ for all $x \in [a,b]$.

or, $-\frac{\epsilon}{4(b-a)} < R_K(x) < \frac{\epsilon}{4(b-a)}$ for all $x \in [a,b]$ ---- (I)

$\therefore$ $R_K(x)$ is bounded on $[a,b]$

Let us take a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} R_K(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} R_K(x)$

$r = 1, 2, 3, \ldots, n$.

Then $U(P, R_K) - L(P, R_K)$

$$= \sum_{r=1}^{n} M_r (x_r - x_{r-1}) - \sum_{r=1}^{n} m_r (x_r - x_{r-1})$$

$$= \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1}) \quad \text{--- (II)}$$

Now from (1)

$M_r \leq \frac{\epsilon}{4(b-a)}$ and $m_r \geq \frac{-\epsilon}{4(b-a)}$, $r=1,2,\dots n$.

$\therefore M_r - m_r \leq \frac{\epsilon}{2(b-a)} < \frac{\epsilon}{b-a}$

From (11), $U(P, R_k) - L(P, R_k) < \sum_{r=1}^{n} \frac{\epsilon}{b-a}(x_r - x_{r-1})$

$$= \frac{\epsilon}{b-a}\sum_{r=1}^{n}(x_r - x_{r-1})$$

$$= \frac{\epsilon}{b-a}\cdot(b-a) = \epsilon$$

Therefore the function $R_n(x)$ is integrable on $[a,b]$

Since $R_n(x) = S(x) - S_n(x)$, $x \in [a,b]$.

gives $S(x) = R_n(x) + S_n(x)$ is integrable on $[a,b]$.

Second part: $\{S_n\}$ converges uniformly to $S(x)$ on $[a,b]$. Let $\epsilon > 0$, there exist a natural number K such that

for all $x \in [a,b]$, $|S_n(x) - S(x)| < \frac{\epsilon}{b-a}$, $\forall n \geq K$

Now $\left|\int_a^b \{S_n(x) - S(x)\}dx\right| \leq \int_a^b |S_n(x) - S(x)|dx$

$$\leq \int_a^b \frac{\epsilon}{b-a}dx$$

$$< \frac{\epsilon}{b-a}(b-a) = \epsilon$$

This implies

$$\lim_{n\to\infty}\int_a^b S_n(x)\,dx = \int_a^b S(x)\,dx$$

But $\int_a^b S_n(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \dots + \int_a^b f_n(x)dx$

Since $\lim_{n\to\infty}\int_a^b S_n(x)dx = \int_a^b S(x)dx$,

So the series $\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx$ converges to $\int_a^b S(x)dx$.

Note:

$$\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx = \int_a^b\left\{\sum_{n=1}^{\infty} f_n(x)\,dx\right\}$$

Example: Prove with proper justification, 8 that $\lim_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$.

Ans Let $f_k(x)=\frac{\cos kx}{k(k+1)}$, $k=1,2,3,\ldots$

Then $|f_k(x)|\leq\frac{1}{k(k+1)}$, $k=2,3,\ldots$

Let $M_k=\frac{1}{k(k+1)}\leq\frac{1}{k^2}$ $(k=2,3,\ldots)$

Since $\sum_{k=2}^{\infty}\frac{1}{k^2}$ is convergent series of positive numbers, so $\sum_{k=2}^{\infty} M_k$ is convergent.

Hence by weierstrass's M-test $\sum_{k=2}^{\infty} f_k(x)$ is uniformly convergent for all real x.

Hence $\lim_{x\to 0}\sum_{k=2}^{\infty} f_k(x)=\sum_{k=2}^{\infty}\lim_{x\to 0} f_k(x)$

If $S(x)$ be the sum of the convergent series $\sum_{k=2}^{\infty} f_k(x)$.

we have $\lim_{x\to 0} S(x)=\sum_{k=2}^{\infty}\frac{1}{k(k+1)}$

Let $u_k=\sum_{i=2}^{k}\frac{1}{i(i+1)}=\sum_{i=2}^{k}\left(\frac{1}{i}-\frac{1}{i+2}\right)=\frac{1}{2}-\frac{1}{k+1}$

Since $\lim_{k\to\infty} u_k=\frac{1}{2}$.

Thus $\lim_{x\to 0} S(x)\Rightarrow\lim_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$

Example: Prove that the Series $\sum_{n=1}^{\infty}\{n^2x^2e^{-n^2x^2}-(n-1)^2x^2e^{-(n-1)^2x^2}\}$ is not uniformly convergent on $[0,1]$.

Ans. Let for $n\in N$, $f_n(x)=n^2x^2e^{-n^2x^2}-(n-1)^2x^2e^{-(n-1)^2x^2}$, $x\in[0,1]$ and $S_n(x)=\sum_{k=1}^{n}f_k(x)$, $x\in[0,1]$.

Then $S_n(x)=f_1(x)+f_2(x)+f_3(x)+\cdots+f_n(x)$

$$=(x^2e^{-x^2}-0)+(2^2x^2e^{-2^2x^2}-x^2e^{-x^2})$$
$$+(3^2x^2e^{-3^2x^2}-2^2x^2e^{-2^2x^2})+\cdots+(n^2x^2e^{-n^2x^2}-(n-1)^2x^2e^{-(n-1)^2x^2})$$
$$=n^2x^2e^{-n^2x^2},\ x\in[0,1],\ n\in N.$$

Now $S_n(0)=0 \Rightarrow \lim_{n\to\infty}S_n(0)=0$ and for $0<x\le 1$, $\lim_{n\to\infty}S_n(x)=0$

Thus $S(x)=\lim_{n\to\infty}S_n(x)=0$ for all $x\in[0,1]$

$\therefore$ The Sequence $\{S_n(x)\}$ converge point-wise to $S(x)$ on $[0,1]$.

Now $M_n=\sup_{x\in[0,1]}|S_n(x)-S(x)|$

$$=\sup_{x\in[0,1]} n^2x^2e^{-n^2x^2}$$

Let $g(x)=n^2x^2e^{-n^2x^2}$

$$g'(x)=2n^2xe^{-n^2x^2}-2n^4x^3e^{-n^2x^2}$$
$$=2n^2xe^{-n^2x^2}(1-n^2x^2)$$
$$g''(x)=2n^2(1-3n^2x^2)e^{-n^2x^2}-4n^4x^2(1-n^2x^2)e^{-n^2x^2}$$
$$=2n^2(1-5n^2x^2+2n^4x^4)e^{-n^2x^2}$$

For Maximum value of $g(x)$ we put $g'(x)=0$

i.e., $1-n^2x^2=0$ or $x=0$.

i.e., $x=\frac{1}{n}$ or $x=0$, $\because x\in[0,1]$

For $x=\frac{1}{n}$, $g''(x)=-4n^2e^{-1}<0$

$\therefore g(x)$ is maximum at $x=\frac{1}{n}$ and the maximum value is $\frac{1}{e}$.

Hence $M_n=\frac{1}{e}$ for all $n\in N$

$\Rightarrow \lim_{n\to\infty}M_n\neq 0$.

Thus $\{S_n\}$ does not converge uniformly to $S(x)$.

$\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ is not uniformly convergent on $[0,1]$, although the sum function $S(x)$ is continuous on $[0,1]$.

Example: Let $f_n(x) = nx^2e^{-nx^3} - (n-1)x^2e^{-(n-1)x^3}$, $x \in [0,1]$. Show that $\sum_{n=1}^{\infty}\int_0^1 f_n(x)dx \neq \int_0^1\left(\sum_{n=1}^{\infty} f_n(x)\right)dx$

Commet by the uniform convergence of the series $\sum_{n=1}^{\infty} f_n(x)$ on $[0,1]$.

Ans: Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$, $x \in [0,1]$, $n \in N$

Then $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

$$= (x^2e^{-x^3} - 0) + (2x^2e^{-2x^3} - x^2e^{-x^3}) + \cdots + (nx^2e^{-nx^3} - (n-1)x^2e^{-(n-1)x^3})$$

$$= nx^2e^{-nx^3}, \quad x \in [0,1], \; n \in N.$$

Now for $x=0$, $S_n(0) = 0$

for $x \in (0,1]$, $e^{nx^3} = 1 + nx^3 + \frac{n^2x^6}{\lfloor 2} + \cdots$

or, $e^{nx^3} > \frac{n^2x^6}{2}$

or, $e^{-nx^3} < \frac{2}{n^2x^6}$

$\therefore \; 0 < S_n(x) < \frac{2}{nx^4}$, $x \in (0,1]$

Now by Sandwich theorem

If theree sequence $\{u_n\}, \{v_n\}, \{w_n\}$ such that $u_n < v_n < w_n$ for all $n \geq m$ $(m \in N)$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = l$

then $\lim_{n\to\infty} v_n = l$.

As $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{2}{nx^4} = 0$

$\therefore \lim_{n\to\infty} S_n(x) = 0$, $x \in (0,1]$.

Hence $\{S_n\}$ converges to the function $S(x)$ where $S(x) = 0$, $x \in [0,1]$ which implies that $\sum f_n(x)$ converges to $S(x)$ on $[0,1]$

Since each $f_n(x)$ being continuous on $[0,1]$ is integrable on $[0,1]$ and

$$\int_0^1 f_n(x)\,dx = \int_0^1 \{nx^2 e^{-nx^3} - (n-1)x^2 e^{-(n-1)x^3}\}\,dx$$

$$= \left[\frac{1}{3}\cdot\frac{e^{-nx^3}}{-1} - \frac{1}{3}\,\frac{e^{-(n-1)x^3}}{-1}\right]_0^1$$

$$= \frac{1}{3}\left(e^{-(n-1)} - e^{-n}\right),\ n \in \mathbb{N}.$$

Let $t_n = \int_0^1 f_1(x)\,dx + \int_0^1 f_2(x)\,dx + \cdots + \int_0^1 f_n(x)\,dx$

$$= \frac{1}{3}(1-e^{-1}) + \frac{1}{3}(e^{-1}-e^{-2}) + \cdots + \frac{1}{3}(e^{-(n-1)} - e^{-n})$$

$$= \frac{1}{3}(1-e^{-n})$$

and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{1}{3}(1-e^{-n}) = \frac{1}{3}$.

Thus $\sum_{n=1}^{\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} t_n = \frac{1}{3}$

Again $\int_0^1 S(x)\,dx = 0$

i.e., $\int_0^1 \sum_{n=1}^{\infty} f_n(x)\,dx = 0$

This proves that $\sum_{n=1}^{\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx$

The series $\sum_{n=1}^{\infty} f_n(x)$ is not uniformly convergent on $[0,1]$, since otherwise,

$$\sum_{n=1}^{\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx.$$

Theorem: (Term-by-Term Differentiation)

Let $[a,b]$ be a closed and bounded interval and for each $n\in\mathbb{N}$, let f_n be differentiable on $[a,b]$. If each f_n' be continuous on $[a,b]$ and the series of functions $f_1'+f_2'+f_3'+\cdots$ converges uniformly on $[a,b]$ to a function g and the series $f_1+f_2+f_3+\cdots$ converges to S on $[a,b]$ then $S'(x)=g(x)$ for all $x\in[a,b]$.

Proof: Let $S_n(x)=f_1(x)+f_2(x)+\cdots+f_n(x)$, $x\in[a,b]$

Since each $f_n(x)$ is differentiable on $[a,b]$ Therefore each $S_n(x)$ is differentiable on $[a,b]$ and $S_n'(x)=f_1'(x)+f_2'(x)+\cdots+f_n'(x)$.

Since each $f_n'(x)$ is continuous on $[a,b]$ then each $S_n'(x)$ is continuous on $[a,b]$.

Since the series $\sum f_n'(x)$ is uniformly convergent to $g(x)$ on $[a,b]$, then the sequence $\{S_n'(x)\}$ converges uniformly to $g(x)$ on $[a,b]$.

For $\epsilon>0$, there exists a natural number K such that,
for all $x\in[a,b]$, $|S_n'(x)-g(x)|<\frac{\epsilon}{b-a}$ for all $n\geq K$.

Since $S_n'(x)$ is continuous on $[a,b]$, then $g(x)$ is continuous on $[a,b]$.

Therefore each $S_n'(x)$ is integrable on $[a,b]$ and $g(x)$ is also integrable on $[a,b]$.

Now $\left|\int_a^b\{S_n'(x)-g(x)\}\,dx\right|\leq\int_a^b|S_n'(x)-g(x)|\,dx$

$$\leq\int_a^b\frac{\epsilon}{b-a}\,dx=\epsilon$$

or $\left|\int_a^b S_n'(x)\,dx-\int_a^b g(x)\,dx\right|<\epsilon$ for all $n\geq K$.

This implies $\lim_{n\to\infty}\int_a^b S_n'(x)\,dx=\int_a^b g(x)\,dx$

$\therefore$ For each $x\in[a,b]$, $\lim_{n\to\infty}\int_a^x S_n'(x)\,dx=\int_a^x g(x)\,dx$ ---(1)

Now by the fundamental theorem 13

$$\int_a^x S_n'(x)\,dx = S_n(x) - S_n(a)$$

Therefore $\lim_{n\to\infty} \int_a^x S_n'(x)\,dx = S(x) - S(a)$.

From (1) $S(x) - S(a) = \int_a^x g(x)\,dx$

Since $g(x)$ is continuous on $[a,b]$, then for all $x \in [a,b]$, $S'(x) = g(x)$.