E-Learning Materials

Sem-2, CC-03, Unit-1

Topic - Real Analysis. (Sets in $\mathbb{R}$)

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15. Show that a finite set has no limit point.

let $S$ be a finite set.

let $S = \{x_1, x_2, \dots, x_m\}$

If possible let $p (\in \mathbb{R})$ be a limit point of $S$.

Then every neighbourhood of $p$ contains infinitely many elements of $S$.

But $S$ is a finite set.

Therefore a neighbourhood of $p$ cannot contain infinitely many element of $S$.

So, $p$ cannot be a limit point of $S$.

Therefore $S$ has no limit point i.e $S' = \phi$.

16. Show that the set of natural numbers has no limit point.

let $p \in \mathbb{R}$.

let $\varepsilon = \frac{1}{2}$

Then the neighbourhood $N(p, \frac{1}{2})$ of $p$ contains at most one natural number namely $p$.

Therefore $p$ cannot be a limit point of $\mathbb{N}$.

So, $\mathbb{N}$ has no limit point.

17. let $S$ be a non empty subset of $\mathbb{R}$, bounded above and $M = \sup S$. If $M \notin S$ prove that $M$ is a limit point of $S$.

let $\varepsilon > 0$. Since $M = \sup S$, we have

i) $x \in S \Rightarrow x < M$ [Since $M \notin S$]

ii) There exists an element $y$ in $S$ s.t $M - \varepsilon < y < M$

$M - \varepsilon < y < M$

or, $M - \varepsilon < y < M < M + \varepsilon$

or, $M - \varepsilon < y < M + \varepsilon$

or, $y \in (M - \varepsilon, M + \varepsilon)$

This shows that the $\varepsilon$-neighbourhood of $M$ contains an element $y$ in $S$.

Since $\varepsilon > 0$ is arbitrary, every neighbourhood of $M$ contains infitely many element of $S$.

Therefore $M$ is a limit point of $S$.

18. show that 0 is a limit point of the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

let us choose $\varepsilon > 0$

By Archimedian property of $\mathbb{R}$, there exists a natural number $n$ such that $0 < \frac{1}{n} < \varepsilon$

or, $-\varepsilon < 0 < \frac{1}{n} < \varepsilon$

or, $-\varepsilon < \frac{1}{n} < \varepsilon$

or, $0 - \varepsilon < \frac{1}{n} < \varepsilon + 0$

or, $\frac{1}{n} \in (0 - \varepsilon, 0 + \varepsilon)$

Since $\varepsilon>0$ is arbitary, every $\varepsilon$ neighbourhood of 0 cont $\frac{1}{n}$ of S. ②

So. 0 is a limit point of S.

**Note:** No other point is a limit point of S. Therefore 0 is the only limit point of S.

Therefore $S'=\{0\}$

19. Let $S=\left\{\frac{1}{m}+\frac{1}{n} : m,n\in\mathbb{N}\right\}$ find $S'$

Let $\varepsilon>0$, by Archimedian property of $\mathbb{R}$, there exists a natural number p such that $0<\frac{1}{p}<\frac{\varepsilon}{2}$

Again by Archimedian property of $\mathbb{R}$, there exists a natural number q such that $0<\frac{1}{q}<\frac{\varepsilon}{2}$

or, $0<\frac{1}{p}+\frac{1}{q}<\varepsilon$

or, $-\varepsilon<0<\frac{1}{p}+\frac{1}{q}<\varepsilon$

or, $-\varepsilon<\frac{1}{p}+\frac{1}{q}<\varepsilon$

or, $0-\varepsilon<\frac{1}{p}+\frac{1}{q}<0+\varepsilon$

or, $\frac{1}{p}+\frac{1}{q}\in(0-\varepsilon,0+\varepsilon)$

Since $\varepsilon>0$ is arbitrary, the neighbourhood $(0-\varepsilon, 0+\varepsilon)$ contains infinitely many elements of S.

So 0 is a limit point of S.

Again for a given $\varepsilon>0$ by Archimedian property of $\mathbb{R}$, there exists a natural number s such that $0<\frac{1}{s}<\varepsilon$

or, $-\varepsilon<\frac{1}{s}<\varepsilon$

or, $\frac{1}{k}-\varepsilon<\frac{1}{k}+\frac{1}{s}<\frac{1}{k}+\varepsilon$

or, $\frac{1}{k}+\frac{1}{s}\in\left(\frac{1}{k}-\varepsilon,\frac{1}{k}+\varepsilon\right)$

Since $\varepsilon>0$ is arbitrary, the neighbourhood $\left(\frac{1}{k}-\varepsilon, \frac{1}{k}+\varepsilon\right)$ contains infinitely many elements of S.

So, $\frac{1}{k}$ is a limit point of S.

No other point is a limit point of S.

Therefore $S'=\left\{0,\frac{1}{k} : k\in\mathbb{N}\right\}$

**Derived set:** Let $S\subset\mathbb{R}$. The set of all limit points of S is said to the derived set and is donted by $S'$.

20. Let A, B be subsets of $\mathbb{R}$ and $A\subset B$. Then $A'\subset B'$.

case:1 $A'=\emptyset$. Then $A'\subset B'$

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Case 2: $A' \neq \emptyset$. Let $p \in A'$. Then $p$ is a limit point of $A$. (3)

Let $\varepsilon > 0$. Then $N(p, \varepsilon)$ contains a point of $A$, say $q$, other than $p$.
$q \in A \Rightarrow q \in B$. Therefore $N'(p, \varepsilon)$ contains a point $q$ in $B$.

Since $\varepsilon$ is arbitrary, $p$ is a limit point of $B$. Therefore $p \in B'$. Thus $p \in A' \Rightarrow p \in B'$ and therefore $A' \subset B'$.

This completes the proof.

21. Let $A \subset \mathbb{R}$. Then $(A')' \subset A'$

<u>case-1:</u> $(A')' = \emptyset$. Then $(A')' \subset A'$

<u>case-2:</u> $(A')' \neq \emptyset$. Let $p \in (A')'$. Then $p$ is a limit point of $A'$.
Let $\varepsilon > 0$. Then $N(p, \varepsilon)$ contains a point of $A'$, say $q$, other than $p$.
Since $q \in A'$, $q$ is a limit point of $A$. Therefore $N(p, \varepsilon)$ being a neighbourhood of $q$ also, contains infinitely many points of $A$.

Since $N(p, \varepsilon)$ contains infinitely many points of $A$, $p$ is a limit point of $A$. That is $p \in A'$

Thus $p \in (A')' \Rightarrow p \in A'$ and therefore $(A')' \subset A'$

22. Let $A, B \subset \mathbb{R}$. Then $(A \cap B)' \subset A' \cap B'$

$A \cap B \subset A \Rightarrow (A \cap B)' \subset A'$, Since $A \subset B \Rightarrow A' \subset B'$
$A \cap B \subset B \Rightarrow (A \cap B)' \subset B'$, Since $A \subset B \Rightarrow A' \subset B'$

It follows that $(A \cap B)' \subset A' \cap B'$

<u>Note:</u> $(A \cap B)' \neq A' \cap B'$ in general.

For example, let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $B = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$
Then $A' = \{0\}$, $B' = \{0\}$, $A \cap B = \{0\}$. $A' \cap B' = \{0\}$, but $(A \cap B)' = \emptyset$.

23. Let $A$ and $B$ be subsets of $\mathbb{R}$. Then $(A \cup B)' = A' \cup B'$

$A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$, Since $A \subset B \Rightarrow A' \subset B'$
$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$, Since $A \subset B \Rightarrow A' \subset B'$

It follows that $A' \cup B' \subset (A \cup B)'$ — (i)

we now prove that $(A \cup B)' \subset A' \cup B'$

Let $p \notin A' \cup B'$. Then $p \notin A'$ and $p \notin B'$

So, there exists a positive $\varepsilon_1$ such that $N'(p, \varepsilon_1) \cap A = \emptyset$ and there exists a possitive $\varepsilon_2$ Such that $N'(p, \varepsilon_2) \cap B = \emptyset$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $\varepsilon > 0$ and $N'(p, \varepsilon) \cap A = \emptyset$, $N'(p, \varepsilon) \cap B = \emptyset$

Therefore $N'(p, \varepsilon) \cap (A \cup B) = [N'(p, \varepsilon) \cap A] \cup [N'(p, \varepsilon) \cap B] = \emptyset$

This disallows $p$ to be a limit point of $A \cup B$. So, $p \notin (A \cup B)'$.

Thus $p \notin A' \cup B' \Rightarrow p \notin (A \cup B)'$

contrapossitively, $p \in (A \cup B)' \Rightarrow p \in A' \cup B'$

consequently, $(A \cup B)' \subset A' \cup B'$ — (ii)